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Minimizing the mass of the codimension-two skeleton of a convex, volume-one polyhedral region

by

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Abstract

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In this paper we establish the existence and partial regularity of a (d-2)-dimensional edge-length minimizing polyhedron in $\mathbb{R}^d$. The minimizer is a generalized convex polytope of volume one which is the limit of a minimizing sequence of polytopes converging in the Hausdorff metric. We show that the (d-2)-dimensional edge-length ζ_{d-2} is lower-semicontinuous under this sequential convergence. Here the edge set of the limit generalized polytope is a closed subset of the boundary whose complement in the boundary consists of countably many relatively open planar regions.

Acknowledgments

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For my parents.

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Chapter 1

Background & Preliminaries

In this chapter, we present our main results, an informal introduction to polyhedral isoperimetric ratios, some history and previous results, and terminology and preliminary theorems used throughout the paper.

1.1 Main Results

For $0 \leq k \leq d$, let $\mathcal{H}^k$ denote the k-dimensional Hausdorff (outer) measure on $\mathbb{R}^d$. Let $\mathcal{P}$ be the family of d-dimensional (convex) polytopes T in $\mathbb{R}^d$ with volume $\mathcal{H}^d(T) = 1$. Let $\sigma_k(T)$ denote the k-dimensional skeleton for any polytope $T \in \mathcal{P}$.

Theorem 1.1 (Main theorem 1). *Let*

$$\mathcal{P} = \{\textit{d-polytopes } T \subset \mathbb{R}^d : \mathcal{H}^d(T) = 1\}.$$

Then there exists a sequence $\{P_i\}_{i=1}^{\infty} \subset \mathcal{P}$ *converging to a convex set* $P \subset \mathbb{R}^d$, *and constants* $0 < r(d) < R(d)$ *such that*

(i). $\bar{B}_{r(d)} \subset P_i, P \subset B_{R(d)}$

(ii). $\mathcal{H}^d(P) = 1$

(iii). $\lim_{i\to\infty} \mathcal{H}^{d-2}(\sigma_{d-2}(P_i)) = \inf_{T\in\mathcal{P}} \mathcal{H}^{d-2}(\sigma_{d-2}(T))$.

Let P be the convex set from Theorem 1 and let $\mathfrak{X}_{d-1}(P)$ denote the set of points in ∂P which have a hyperplanar neighborhood in ∂P. These may be thought of as the "face points" of ∂P and their complementary "edge set" $\partial P \setminus \mathfrak{X}_{d-1}(P)$ satisfies

Theorem 1.2 (Main Theorem 2)**.** $\mathcal{H}^{d-2}(\partial P \setminus \mathfrak{X}_{d-1}(P)) < \infty$.

Now let $\mathfrak{X}_{d-2}(P)$ denote the set of "good edge" points in the edge set which have a neighborhood in ∂P lying in two transverse hyperplanes. Then

Theorem 1.3 (Main Theorem 3)**.** $\mathcal{H}^{d-2}(\partial P \setminus (\mathfrak{X}_{d-1}(P) \cup \mathfrak{X}_{d-2}(P))) = 0$.

Thus, $\mathcal{H}^{d-2}$ almost all of the points in the boundary of the limiting set P are either face points or good edge points. Finally, we show that the $\mathcal{H}^{d-2}$ measure of the edge sets is lower-semicontinuous in this limit with respect to the Hausdorff metric:

Theorem 1.4 (Main Theorem 4)**.** $\mathcal{H}^{d-2}(\sigma_{d-2}(P)) \leq \liminf \mathcal{H}^{d-2}(\sigma_{d-2}(P_i))$.

The proofs for Theorem 1 utilize the Blaschke selection theorem and the fact that bounded edge length coupled with the unit-volume requirement for a polytope provides a natural uniform bound for the polytope. For Theorems 2 and 3, the boundaries for the sets are expressed as graphs of Lipschitz functions whose gradients are contained in the space $[SBV]^{d-1}$. A pseudocompactness theorem for SBV functions is used to show regularity for the limiting function. The strategy for the proof of Theorem 4 uses Caccioppoli partitions and BV functions on a manifold. The projection of the faces of the polytopes define a partition of the sphere and then we apply the compactness of Caccioppoli partitions. The faces of the polytopes can then be viewed as currents converging in the flat norm. The lower-semicontinuity follows from the containment of the "edge set" of the limiting convex set P in the image of the limiting Caccioppoli partition.

1.2 Introduction

Isoperimetric problems have been studied since ancient times. The most famous of these states that the most volume that can be enclosed with a fixed amount of boundary is achieved by the ball. Polyhedral isoperimetric problems allow a more generalized version of this problem to be stated. For polyhedra (let us assume for now in $\mathbb{R}^3$) we have a well-defined notion of faces, edges, and vertices. Thus, we can ask the question if there exists a polyhedron in $\mathbb{R}^3$ which minimizes ratios of any combination of edge length, area of faces, or number of vertices to each other. More precisely, if $P \subset \mathbb{R}^3$ and A, E and N are the area of its faces, total length of its edges, and number of vertices, respectively, then we could ask the question, for any nonnegative constants α, β, γ, and λ, if there exists a polyhedron in $\mathbb{R}^3$ which minimizes the weighted combination of ratios

$$\mathcal{I} = \alpha N^3 + \beta \frac{L^3}{V} + \gamma \frac{A^3}{V^2} + \lambda \frac{L^2}{A}.$$

The exponents ensure that the ratios depend purely on the geometry of the polyhedron and not on the scaling. In this paper, we are interested only in the case when $\alpha = \gamma = \lambda = 0$. In general, in $\mathbb{R}^d$ we will be minimizing

$$\frac{L^d}{V^{d-2}}$$

where L is the (d-2)-dimensional edge length and V is the d-dimensional volume. Thus we are trying to minimize the (d-2)-dimensional edge-length with fixed d-dimensional volume.

1.3 Summary

In the first chapter of this paper we introduce the idea of polyhedral isoperimetric problems and give a brief overview of work that has already been done in the area. We introduce Melzak's conjecture and other problems of interest in $\mathbb{R}^3$. We then give a brief summary of the relevant work done by Eggleston, Grünbaum, and Klee for higher dimensional isoperimetric problems. We also present a summary of the relevant work in Berger's Ph.D. thesis concerning Melzak's conjecture and limits of polyhedral objects. Finally, we present preliminary terminology and theorems that will be used in the paper.

In the second chapter we show the existence of a convex object that is the limit of a sequence of d-dimensional volume-one polytopes minimizing (d-2)-dimensional edge-length. To do this, we decompose the boundary of the limiting function into finitely many portions and express the boundary as the graphs of Lipschitz functions.

In the third chapter, we show that the $\mathcal{H}^{d-2}$ measure of the points in the boundary of the limiting convex set which are not "face points" is finite. We also show that $\mathcal{H}^{d-2}$ almost all the points in the boundary of the limiting convex set must either be face points or "good edge points". This is achieved by showing a lower bound on the density of the discontinuity set for the SBV function whose graph is the boundary of the limiting set.

In the fourth chapter we demonstrate the lower-semicontinuity of the Hausdorff measure of the edge set of the limiting convex set of a sequence of minimizing polytopes. We express the boundaries as graphs of Lipschitz functions over the sphere and the faces of the codimension one skeletons as currents converging in the flat norm.

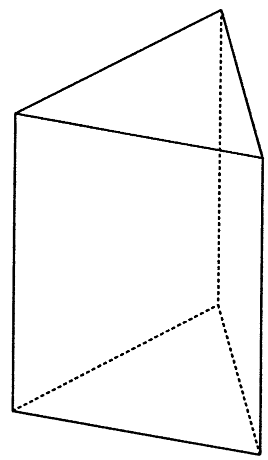

Figure 1.1: The equilateral right triangular prism is the conjectured minimizer for Melzak's problem.

1.4 Melzak's conjecture and related problems

In 1965, Z.A. Melzak wrote a survey paper over open problems relating to convexity. In it, he made the following conjecture:

Conjecture 1.5 ([Mel65]). *Let P be a convex polyhedron in $\mathbb{R}^3$ with volume V and sum of all edge-lengths L. Then*

$$\frac{V}{L^3} \leq \frac{1}{2^2 3^{\frac{11}{2}}}$$

and the equality holds if and only if P is similar to a right prism whose base is an equilateral triangle with side-length equal to the height of the prism.

This conjecture was made in the section of problems whose answers he thought would not be difficult to obtain. Note that he restricted his conjecture to the class of *convex* polyhedra, but the same conjecture may be made for all polyhedra.

He went on to solve for the minimum value when restricting to the class of tetrahedra in $\mathbb{R}^3$:

Theorem 1.6 ([Mel66]). *Let T be a tetrahedron and let $V(T)$ and $L(T)$ denote its* *volume and the sum of its edge-lengths. Then*

$$\frac{V(T)}{L^3(T)} \leq \frac{1}{6^4 2^{\frac{1}{2}}}$$

with equality if and only if the tetrahedron T is regular.

A regular tetrahedron is a tetrahedron with equal length sides and equal area faces. He utilized only elementary methods of calculus and geometry for his proof.

Related results

Similar results have been found when the class of polyhedra is restricted. In 1948, L. Fejes Tóth conjectured that, when restricting to the class of polyhedra which contain the unit sphere, a minimizer was the unit cube [Tót48][BE57]. Or, equivalently,

Conjecture 1.7 ([Tót48]). *Let P be a polyhedron that contains the unit sphere and let L be the total edge length of P. Then $L \geq 24$.*

Hammersley obtained partial results by restricting this problem further by uniformly bounding the number of edges each face contained:

Theorem 1.8 ([Ham51]). *Let P be a polyhedron which contains the sphere of unit radius, let L be its total edge length, and assume no face is a polygon of more than n sides. Then*

$$L > \frac{20}{3}\sqrt{\pi n \tan\left(\frac{\pi}{n}\right)}$$

We note that the actual bound Hammersley obtained in his paper is half this value as he was considering the sphere of unit *diameter*. Hammersley also states that Tóth found the bound $L > 10$ for all polyhedra and $L > 14$ for polyhedra with only triangular faces.

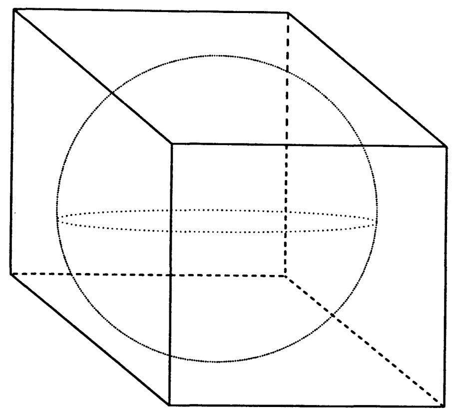

Figure 1.2: The cube is the unique minimizer of $\mathcal{I}$ for polyhedra which contain the sphere.

Besicovitch and Eggleston went on to prove Tóth's original conjecture and in addition proved that the cube was the unique minimizer among polyhedra which contained the unit sphere [BE57]. Again, their proof was geometric in nature.

Besicovitch continued to study a related problem [Bes63]. He defined a *crate* as a three-dimensional object which can be realized as the edge set of a convex polyhedron. He then asked the question of what could be proved about the edge length of a crate which contained the unit sphere without allowing it to slide out. Note that the crate must only contain the unit sphere, not the polyhedron spanned by the crate, which allows the sphere to "poke out" of the faces. Besicovitch constructed a sequence of crates containing the sphere whose edge lengths were converging to the value

$$\gamma = \frac{8}{3}\pi + 2\sqrt{3}.$$

Note that $\gamma \approx 11.88 < 24$ because the sphere is allowed to poke out of the faces. Aberth then proved the following theorem:

Theorem 1.9 ([Abe63]). *Let C be a crate with contains the unit sphere which cannot* *slide out and let L be the total length of its edges. Then*

$$L > \gamma.$$

Combined with Besicovitch's result, this proved that there does not exist a polyhedral minimizer to the crate problem.

In order to prove his theorem, Aberth proves a lemma that was later utilized by Berger in his Ph.D. thesis [Ber02]. It is another variant of an isoperimetric bound for convex polyhedra:

Lemma 1.10. *For any convex polyhedron with total surface area A and total edge length L, we have*

$$A < \frac{L^2}{6\pi}.$$

In a later paper, Aberth proves a lower bound for the ratio $\mathcal{I}$:

Theorem 1.11 ([Abe73]). *Let P be a polyhedron and let L, V be the total edge length of the polyhedron and its volume, respectively. Then*

$$\mathcal{I} = \frac{L^3}{V} > 2\pi \cdot 6^3.$$

Therefore, for polyhedra with volume one, the total edge length is bounded below: $L > 6(2\pi)^{\frac{1}{3}} \approx 11.1$ and the conjectured minimizer, the right triangular prism, has $L = 2^{\frac{2}{3}} \cdot 3^{\frac{11}{6}} \approx 11.9$. The lower bound guarantees that the ratio cannot degenerate to zero in the limit, i.e. that we cannot get a finite volume with an infinitesimal edge length.

1.5 Poyhedral isoperimetric ratios in higher dimensions

Isoperimetric problems can be generalized to higher dimensions. Higher dimensional convex polyhedra are called (convex) polytopes and the union of all s-dimensional faces ($s \leq d$) of a polytope P is called the s-skeleton of P.

Definition 1.12. Let the $\mathcal{H}^s$-measure of the s-skeleton of a polytope P be defined by $\zeta_s(P)$.

These definitions will be made more precise later in the paper. Following [Kle70], one can define the isoperimetric ratio

$$\rho_{r,s}(P) = \frac{\zeta_s(P)^{\frac{1}{s}}}{\zeta_r(P)^{\frac{1}{r}}}$$

for any d-polytope P. We may then ask whether the ratio is bounded in the class of d-polytopes, i.e. whether

$$\gamma(d, r, s) = \sup_{\mathcal{P}_d}\{\rho_{r,s}(P)\}$$

is finite, where $\mathcal{P}_d$ is the set of all d-polytopes. In the case when this ratio is bounded, we may further inquire:

1. Does there exist a d-polytope that achieves this maximum?
2. If yes, is this polytope unique?

For example, the classical isoperimetric problem tells us that

$$\gamma(d, d-1, d) = (d\omega_d^{\frac{1}{d}})^{\frac{1}{d-1}}$$

as we are able approximate the d-ball by d-polytopes, but a polytope does not achieve the desired bound. Melzak's conjecture [Mel65] can be restated that $\gamma(3, 1, 3) =$

$2^{-\frac{2}{3}} \cdot 3^{-\frac{11}{6}}$ and the only polytope that can achieve this bound is the equilateral right triangle whose side-length is equal to its height. Aberth's inequality [Abe63] proved that $\gamma(3,1,2) \leq (6\pi)^{-\frac{1}{2}}$ and Kömhoff proved that $\gamma(3,1,2) \geq (\frac{16}{3}\pi+2\sqrt{3})^{-\frac{1}{2}}$ [Köm70]. Klee proved that for all d, if $s < r$, then $\gamma(d,r,s)$ is infinite. He also conjectured the converse [Kle70].

Eggleston *et. al.* proved the finiteness of γ in some special cases:

Theorem 1.13 ([EGK64]). *Let $1 \leq r \leq s \leq d$ and let $s = d$ or $s = d-1$. Then $\gamma(d,r,s)$ is finite.*

They also use the fact that convex polytopes are dense in the space of compact convex subsets of $\mathbb{R}^d$ to extend the definition of ζ_s to the set of compact convex bodies. They use this definition to derive compact convex sets that achieve a maximum values of γ under suitable hypotheses, but unfortunately this definition has no obvious geometric interpretation as we will show later (see Figure 1.6). Under certain conditions, they are able to obtain a polytope that achieves a minimum value for ζ_r among polytopes with a uniformly bounded number of t-faces for certain values of t:

Theorem 1.14. *Let $f_t(P)$ be the number of t-faces of P. Suppose r, t, d, k are integers with $1 \leq r < d$, $0 \leq t < d$, and $k \geq \binom{d+1}{t+1}$. Then among the d-polytopes P of unit volume for which $f_t(P) \leq k$, there are those for which $\zeta_r(P)$ is a minimum.*

Thus, uniformly bounding the number of t-faces for any integer $t \in [0, d-1]$ is sufficient to guarantee the existence of a *polyhedral* minimizer of ζ_r.

Finally, Lillington [Lil74] studied n-dimensional simplicial polytopes that could be inscribed in a sphere of radius λ. So instead of bounding the polyhedra from the inside, he forces the vertices of an n-simplex to be located on a sphere of radius λ. He then proved that any d-dimensional simplex inscribed in the sphere of radius λ

Platonic Solid	$\mathcal{I}$
Tetrahedron	$1296\sqrt{2}$
Cube	1728
Octahedron	$2592\sqrt{2}$
Dodecahedron	$172800\sqrt{5}(1+\sqrt{5})^{-4}$
Icosahedron	$129600(1+\sqrt{5})^{-2}$

Figure 1.3: [Ber02] Calculated values for $\mathcal{I}$ for the platonic solids.

must satisfy the inequality

$$\zeta_1(P) > 2d\lambda.$$

1.6 Berger's thesis

Berger has done, up to now, the most extensive investigation into two polyhedral isoperimetric ratios in $\mathbb{R}^3$ in his Ph.D. thesis, one of them being the ratio considered by Melzak. Again, let

$$\mathcal{I} = \frac{L^3}{V}$$

where L and V are the total edge length and volume of a polyhedral region, respectively. He calculated the value $\mathcal{I}$ for the convex regular polyhedrons, commonly referred to as the platonic solids. His results showed that, among the platonic solids, the cube minimized the ratio $\mathcal{I}$ (see Figure 1.3).

Berger also found the minimizer among prisms. He defined a prism as follows:

Definition 1.15. We call $P \subset \mathbb{R}^3$ a prism if there exists a polygon $B \subset \mathbb{R}^2 \subset \mathbb{R}^3$ and a vector v (not necessarily perpendicular to the plane that contains B) such that

$$P = \{tvx : t \in [0,1], x \in B\}$$

Berger then proved the following theorem:

Theorem 1.16 ([Ber02]). *The right regular triangular prism minimizes $\mathcal{I}$ in the class of all prisms.*

The proof is a result of the fact that any prism can be made into a right prism without changing edge length, while the volume strictly increases. In addition, for any right prism with an n-gon as a base, the prism with a regular n-gon as a base with equal perimeter and equal height will have the same edge length and strictly greater volume. Finally, it is easily shown that that the right triangular prism minimizes $\mathcal{I}$ among the right prisms. Berger also proved that the right regular tetrahedron minimizes $\mathcal{I}$ for all right regular pyramids. We note that none of Berger's results disprove Melzak's original conjecture.

Berger's main result of his thesis was to show the existence of a minimizer in a larger class of convex regions and regularity properties of these sets. For any convex set C, he defined the following:

Definition 1.17. A point $p \in \partial C$ will be called an *face point* if there exists neighborhood $p \in U$ and a plane P such that $U \cap \partial C \subset P$. The set of face points are denoted F. *Edge points* are defined as elements of the set $E := \partial C \setminus F$.

Berger proved the main theorem about the existence of a minimizer among convex polyhedral regions of $\mathcal{H}^3$-measure one:

Theorem 1.18 ([Ber02]). *Let*

$$\mathcal{P} := \{\textit{convex polyhedra } P \subset \mathbb{R}^3 : \mathcal{H}^3(P) = 1\}$$

and let E_P be the associated edge set of each polyhedron $P \in \mathcal{P}$. Let $\{P_i\} \subset \mathcal{P}$ such that

$$\lim_{i\to\infty} \mathcal{H}^1(E_{P_i}) = m := \inf_{P\in\mathcal{P}} E_P.$$

Then P_i converges with respect to the Hausdorff metric to a convex set C such that $\mathcal{H}^3(C) = 1$ and $\mathcal{H}^1(E_C) \leq m$.

His proof relies on the lower-semicontinuity of $\mathcal{H}^1$ for connected sets with respect to the Hausdorff metric. Berger goes on to show that the set of "bad edge points" has $\mathcal{H}^1$-measure zero.

Definition 1.19. Let C be a convex set and let E_C be the edge set associated to C. We say that a point $p \in E_C$ is a *good edge point* if there exists a neighborhood $p \in U$ and planes P_1, P_2 such that $U \cap \partial C \subset P_1 \cup P_2$. Let G_C be the set of good edge points. We say a point $p \in E_C$ is a *bad edge point* if $p \in V_C := E_C \setminus G_C$.

Berger then proves the theorem:

Theorem 1.20. *Let C be the convex set from Theorem 1.18. Then*

$$\mathcal{H}^1(V_C) = 0.$$

1.6.1 Convexity and pathological examples

Note that Melzak's original conjecture was for convex polyhedra, but, due to Berger's definitions for edge set and face set, the same question could be posed for *general* polyhedra. It is still unknown whether the minimizer in this class of polyhedra (if it exists) would have to be convex. To illustrate the difficulties, Berger provided examples of polyhedra in which taking the convex hull would increase the value of $\mathcal{I}$. We provide one below known as the bird's beak (see Figure 1.4). Notice that the angle made between the two tetrahedra may be made wide so that the volume gained in taking the convex hull is arbitrarily small while the edge length being added is bounded from below.

In another part of his thesis, Berger illustrates pathological examples of what can happen in the limit of polyhedra when taken with respect to the Hausdorff metric.

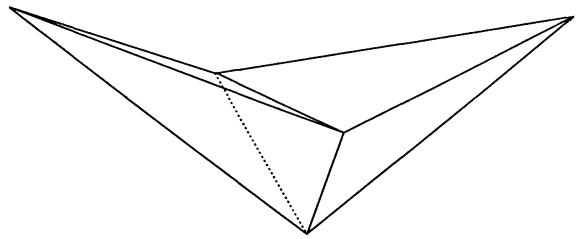

Figure 1.4: Example of a non-convex polyhedron such that $\mathcal{I}(P) < \mathcal{I}(\text{conv}(P))$.

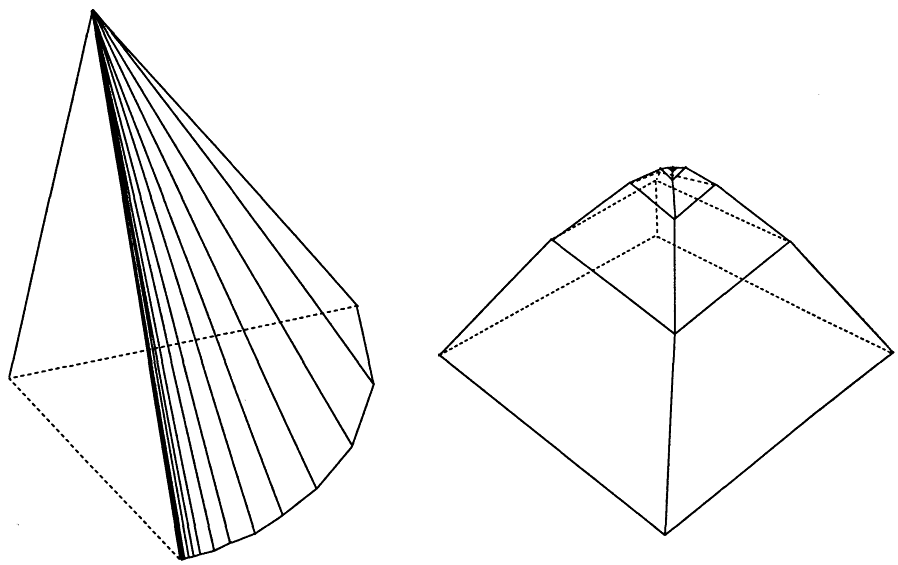

Figure 1.5: Two examples convex sets with infinitely many faces.

We provide two examples in Figure 1.5. The first is an example of a convex set with a point in its boundary that is contained in infinitely faces. The second is an example of a convex set with a point it its boundary that is not contained in any face. Here this is the only such point. By contrast, no point of the sphere is contained in a face because the sphere has none. In particular, Berger's minimizer C in Theorem 1.18 is a "generalized" polyhedron in the sense that

$$\mathcal{H}^2(\partial C \setminus (F_C \cup E_C)) = 0.$$

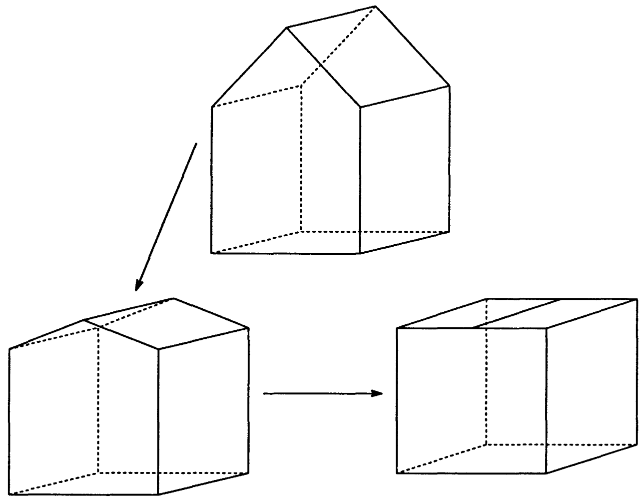

Figure 1.6: Polyhedra converging in the Hausdorff metric, but $L \neq \liminf L_i$.

Berger also provided examples pathological examples of limiting convex polyhedra whose sets of bad edge points did not have $\mathcal{H}^1$ measure zero. They consisted of edge sets with Cantor-properties and refer the reader to [Ber02] for examples.

Finally, we give an example of a sequence of polyhedra converging in the Hausdorff metric, but $L \neq \liminf L_i$ (see Figure 1.6). The extra edge measure disappears in the limit as the "roofs" on the houses collapse. This is an example for why ζ_{d-2} cannot be defined continuously on the closure of the set of polyhedra with respect to the Hausdorff metric. Nevertheless, it is lower-semicontinuous.

1.7 Terminology

1.7.1 Convex Sets

We need to define some structure to be able to define "edge sets" and "faces" of sets in higher dimensions. Let V be a vector space and let $A \subset V$.

Definition 1.21. We say A is *convex* if for every $x, y \in A$ and $\lambda \in (0,1)$ we have

$$\lambda x + (1-\lambda)y \in A$$

Thus, A is convex if, for any two points in A, the line between those two points is also contained in A. We have the following well-known theorem about convex sets:

Theorem 1.22. *Let $\mathcal{F}$ be any family of convex sets. Then*

$$\cap_{C \in \mathcal{F}} C$$

is convex.

Using Theorem 1.22, if a set is not convex, then we have an operation that takes that set to the smallest convex set which contains it.

Definition 1.23. Let $\mathcal{F}_A = \{C \subset X : C \text{ is convex, } A \subset C\}$. Then define the *convex hull of A* as

$$\operatorname{conv}(A) = \cap_{C \in \mathcal{F}_A} C$$

or, equivalently,

$$\operatorname{conv}(A) = \{\lambda x + (1-\lambda)y : x, y \in A, \lambda \in [0,1]\}$$

It is easy to show that the two definitions are equivalent as the second is a convex set that contains A but also contains the minimal set of points necessary to make A convex.

By Theorem 1.22, we know that $\operatorname{conv}(A)$ must be a convex and, by definition, is the smallest convex set that contains A. Note that a subspace, being closed under addition and scalar multiplication, is a convex set.

We now want to have a notion of a subspace of a vector space that does not necessarily contain the origin. For that, we define the following:

Definition 1.24. We say $S \subset X$ is an *affine subspace* if S is of the form $S = x + L$ for some $x \in X$ and some linear subspace $L \subset X$.

Similar to the convex hull of a set A, we may also take the affine hull of any subset $A \subset X$.

Definition 1.25. We define the *affine hull* of A as the intersection of all affine subspaces which contain A, or, equivalently,

$$\text{aff}(A) = \{\lambda x + (1 - \lambda)y : x, y \in A, \lambda \in \mathbb{R}\}.$$

Definition 1.26. Let $A \subset X$ be an affine subspace so that $A = x + L$ for some linear subspace $L \subset X$ and some $x \in X$. Then define the *dimension of* A as the dimension of L:

$$\dim(A) = \dim(L)$$

Because our choice of a point and a linear subspace were arbitrary, it remains to show that this is well-defined.

Proposition 1.27. *The dimension of an affine subspace is well-defined.*

Proof. Let $A = x + L = x' + L'$. Then $L' = (x - x') + L$ and $L = (x' - x) + L'$. Since L is a linear subspace of X, then $0 \in L$ and hence $(x - x') \in L'$. Again, since L' is a linear subspace of X, then $(x' - x) = -(x - x') \in L'$. Therefore, $L \subset L'$. Similarly, $L' \subset L$. Therefore, $L = L'$ and so $\dim(L) = \dim(L')$. □

We can now define the dimension of any convex set in terms of the dimension of its affine hull, which is an affine subspace.

Definition 1.28. Let C be a convex set. Then

$$\dim(C) := \dim(\text{aff}(C)).$$

For a convex set not of full dimension, it is convenient to view it as a subset of the affine hull and consider its boundary and interior as a subset of the affine space with the topology inherited from the metric on the vector space. Thus we have the following definitions:

Definition 1.29. The *relative interior* of a convex set C is the interior of C as a subspace of the affine hull of C and denote it

$$\text{ri}(C).$$

Definition 1.30. We define the *relative boundary* of C to be

$$\text{rb}(C) = \text{cl}(C) \setminus \text{ri}(C).$$

1.7.2 Regular faces

For any arbitrary convex set, we also have a notion for a face of the convex set. First, some preliminary definitions:

Definition 1.31. We define the *line between x and y*, denoted $[x, y]$, as the convex hull of the set $\{x, y\}$, or

$$[x, y] = \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$$

Definition 1.32. We define the *open line segment between x and y*, denoted (x, y), as the $\text{ri}([x, y])$, or

$$(x, y) = \{\lambda x + (1 - \lambda)y : \lambda \in (0, 1)\}$$

We now define a face for any convex set:

Definition 1.33. Let $C \subset X$ be a convex set. We say $F \subset C$ is a *face of* C if F is convex and for every $x, y \in C$ such that $(x, y) \cap F \neq \emptyset$, we have

$$[x, y] \subset F.$$

Remark 1.34. The regular faces of a closed convex set are closed and form a lattice. Also, it can be checked that the intersection of any family of faces is again a face.

For any closed convex set C, let $\mathcal{F}(C)$ be the set of faces for C. We then have the following two useful theorems:

Theorem 1.35 ([Brö83]). *Let C be a closed convex set in $\mathbb{R}^d$, let $x \in C$, and $F \in \mathcal{F}(C)$. Then F is the smallest face of C containing x if and only if $x \in ri(F)$.*

Corollary 1.36 ([Brö83]). *Let C be a closed convex set in $\mathbb{R}^d$ and let $\mathcal{F}(C)$ be the set of faces of C. Then the set*

$$\{ri(F)\}_{F \in \mathcal{F}(C) \setminus \emptyset}$$

forms a partition of C.

1.7.3 Polytopes and exposed faces

We may now define the higher dimensional analog of a convex polyhedron for any vector space X.

Definition 1.37. A *polytope* or *convex polytope* is the the convex hull of a nonempty finite set in X.

We now need to introduce the idea of a "face" and "edge" of a polyhedron in higher dimensional space. Since polytopes have a special structure, we may define a more restrictive version of a face called an exposed face. Again, we have some preliminary definitions.

Let X be a Banach Space over $\mathbb{R}$, and let X^* be the dual space of X.

Definition 1.38. A *closed affine halfspace* $H \subset X$ is defined to be a set of the form

$$H = \{v \in X : f(v) \leq \alpha\}$$

for some $f \in X^*$ and some $\alpha \in \mathbb{R}$.

We also want to keep track of the boundary of the closed affine halfspace, called the affine hyperplane.

Definition 1.39. An *affine hyperplane* $P \subset X$ is defined to be a set of the form

$$P(f, \alpha) = \{v \in X : f(v) = \alpha\}$$

It can be shown that an affine hyperplane is in fact an affine subspace. For example, if $X = \mathbb{R}^d$, then $\dim(P(f, \alpha)) = d - 1$. We state a fact about convex sets that will be useful later.

Fact 1.40 ([Grü03]). *A closed convex set is equal to the intersections of all the closed halfspaces that contain it.*

Now let $K \subset X$ be a closed convex set.

Definition 1.41. We say that the affine hyperplane $P(f, \alpha)$ is a *supporting hyperplane* of K provided $K \subset H(f, \alpha)$ and $P(f, \alpha) \cap K \neq \emptyset$.

Note that $P(f, \alpha) = P(-f, -\alpha)$ and so K can be a subset of $H(f, \alpha)$ or $H(-f, -\alpha)$. This just says that K must lie completely on one side or the other of a supporting hyperplane while also intersecting the hyperplane.

Definition 1.42. We say that $F \subset K$ is a *proper exposed face* of K if there exists a supporting hyperplane $P(f, \alpha)$ such that $F = K \cap P(f, \alpha)$.

Clearly from the definition we see that a proper exposed face of a convex must itself be convex since it is the intersection of two convex sets. Note that a proper exposed face can also be any dimension (i.e. they do not have to be codimension one).

Remark 1.43. It can be checked that an exposed face is also a regular face, but the converse is not necessarily true for general convex sets C.

The converse *is* true for polytopes:

Theorem 1.44 ([Brö83]). *Every face of a polytope P is an exposed face.*

Definition 1.45. We call K and $\emptyset$ the *improper exposed faces* of K. Together with the proper exposed faces they make up the *exposed faces* of K.

Recalling the definition of the dimension of a convex set and the fact that faces are convex, then each face has a dimension and can be grouped accordingly.

Definition 1.46. Let F be a face of K. Then we say F is an *s-face* if

$$\dim(F) := \dim(\text{aff}(F)) = s.$$

We define the *s-skeleton* of K to be the union of all the s-faces of K, which we denote $\sigma_s(K)$ (and we let $\dim(\emptyset) = -1$). We also define the function ζ_s to be the s-dimensional Hausdorff measure of the s-skeleton of a closed convex set, so that $\zeta_s(K) := \mathcal{H}_s(\sigma_s(K))$.

We will find it convenient to look at a coarser division of C by the dimension of the faces, and so we define the following:

Definition 1.47. Let C be a closed convex set and let $\mathfrak{X}_s(C)$ be defined as the union of the relative interiors of all the s-faces, or

$$\mathfrak{X}_s(C) = \cup\{\mathrm{ri}(F) : F \text{ is an s-face of C}\}.$$

Remark 1.48. From Theorem 1.36 and Theorem 1.44, if C is a polytope such that $\dim(C) = d$, then we get that $\{\mathfrak{X}_s(C)\}_{s=0}^{d}$ forms a partition of C, and therefore

$$\{\mathfrak{X}_s(C)\}_{s=0}^{d-1}$$

forms a partition of ∂C because $\mathfrak{X}_d(C) = C \setminus \partial C$.

We also have a lattice structure for the faces, which is a consequence of the following theorem:

Theorem 1.49 ([Grü03]). *Let P be a convex d-polytope and let $-1 \leq h < k \leq d-1$. Then each h-face of P is the intersection of the family (containing at least k-h+1 members) of k-faces of P containing it.*

In particular, we get the useful conclusion for d-polytopes:

Corollary 1.50. *Let P be a convex d-polytope and let $-1 \leq h \leq k \leq d$. Then*

$$\sigma_h(P) \subset \sigma_k(P).$$

We will also use a more specific statement concerning the codimension two faces of a polytope:

Theorem 1.51 ([Grü03]). *Let P be a d-polytope. Then each $(d-2)$-face F of P is contained in precisely two codimension one faces F_1 and F_2 of P, and $F = F_1 \cap F_2$.*

For any *general* closed convex set C we do not have such a neat partition of the boundary as there exists for polytopes, but since exposed faces are regular faces, the

$\mathfrak{X}_s$ still enjoy disjointedness courtesy of Theorem 1.35. A slightly stronger but very similar statement can be made:

Lemma 1.52. *Let $C \subset \mathbb{R}^d$ be any closed convex set. Then*

$$\sigma_s(C) \subset C \setminus (\cup_{s<k\leq d}\mathfrak{X}_k(C))$$

Proof. Suppose not. Let F be an s-face such that $x \in F$ and let G be a k-face such that $x \in \mathrm{ri}(G)$. But since exposed faces are regular faces and the intersection of faces are again faces, then this implies that $x \in F \cap G$ and $F \cap G$ is a face of C. Because

$$\dim(F \cap G) \leq \dim(F) < \dim(G)$$

this implies that $F \cap G \neq G$. Since $F \cap G \subset G$ and $x \in \mathrm{ri}(G)$, this contradicts Theorem 1.35. □

Corollary 1.53. *Let $C \subset \mathbb{R}^d$ and let $s < d-1$. Then*

$$\sigma_s(C) \subset \partial C \setminus (\cup_{s<k\leq(d-1)}\mathfrak{X}_k(C)).$$

Proof. The sets $\{\mathfrak{X}_k(C)\}$ are disjoint and $C \setminus \mathfrak{X}_d(C) = \partial C$. □

For the rest of this paper, our vector space will be $\mathbb{R}^d$ with the standard norm. We note that $(\mathbb{R}^d)^* = \mathbb{R}^d$ where the affine hyperplanes are defined by a vector perpendicular to the plane and a point located on the hyperplane. We will also use the term *face* to refer to *exposed faces.* For polytopes there is obviously no distinction to be made.

Chapter 2

Existence of a minimizer

In this chapter we show that the set of d-polytopes with unit volume is nonempty. We then find an edge-length minimizing sequence of polytopes with unit volume which converges to a convex set in the Hausdorff metric. We do this by showing that each polytope in the sequence is uniformly bounded by a larger ball and contains a smaller ball whose radii only depend on the dimension d. This limiting convex set also has $\mathcal{H}^d$ measure one. We then express portions of the boundaries of the convex sets as graphs of Lipschitz functions, and show that the gradients of these Lipschitz functions are special functions of bounded variation. We prove Main Theorem 1 in this chapter which is found in §2.3.2.

2.1 The hypercube

In this section we will show that the hypercube is a d-polytope with unit volume using the definitions provided in the previous sections.

One technicality that we will encounter is that we want to take an infimum over a set of numbers, and to do so we want that set of numbers to be non-empty. Also, we want a uniform upper bound for a sequence of polytopes that are minimizing ζ_{d-2}.

This section will provide us with both by proving the set

$$\mathcal{P} := \{\text{convex polytopes } P \subset \mathbb{R}^d : \mathcal{H}^d(P) = 1\}$$

is non-empty.

Let $I := [0,1] \subset \mathbb{R}$. We define the hypercube in $\mathbb{R}^d$ as

$$C^d = \underbrace{I \times \cdots \times I}_{\text{d-times}}$$

Note that $\mathcal{H}^d(C^d) = \mathcal{H}^1(I)^d = 1$. Also, let $S = \{\mathbf{x} \in \mathbb{R}^d : x_i \in \{0,1\} \text{ for all } i\}$ Then we see that S has finitely many points and $\mathcal{H}^0(S) = 2^d$. It is also clear that $\text{conv}(S) = C^d$. Therefore, C^d is a convex polytope of unit volume.

Lemma 2.1. $\sigma_{d-1}(C^d) = (\cup_{i=1}^d \mathbb{F}_i^0) \cup (\cup_{i=1}^d \mathbb{F}_i^1)$ *where*

$$\mathbb{F}_i^k := I \times \cdots \times \underbrace{k}_{i^{th}\text{-place}} \times \cdots \times I$$

Proof. By Remark 1.48, we know that $\sigma_{d-1}(C^d) \subset \text{rb}(C^d) = \partial C^d = (\cup_{i=1}^d \mathbb{F}_i^0) \cup (\cup_{i=1}^d \mathbb{F}_i^1)$. We define the affine hyperplanes

$$\mathbb{P}_i^k := \{\mathbf{x} \in \mathbb{R}^d :< \mathbf{x}, e_i >= k\} = \{\mathbf{x} \in \mathbb{R}^d : x_i = k\}$$

where $\{e_i\}_{i=1}^d$ is the standard euclidean basis in $\mathbb{R}^d$. Then it is clear that $\mathbb{P}_i^k \cap C^n = \mathbb{F}_i^k$. Therefore we get

$$(\cup_{i=1}^d \mathbb{F}_i^0) \cup (\cup_{i=1}^d \mathbb{F}_i^1) \subset \sigma_{d-1}(C^d)$$

and we reach our desired conclusion. □

We now have the following lemma similar to the previous determining what the edge set of the hypercube must be:

Lemma 2.2.

$$\sigma_{d-2}(C^d) = (\cup_{i\neq j}\mathbb{E}^0_{ij}) \cup (\cup_{i<j}\mathbb{E}^1_{ij}) \cup (\cup_{i<j}\mathbb{E}^{-1}_{ij})$$

where

$$\mathbb{E}^0_{ij} := I \times \cdots \times \underbrace{0}_{i^{th}\text{-place}} \times \cdots \times \underbrace{1}_{j^{th}\text{-place}} \times \cdots \times I$$

$$\mathbb{E}^1_{ij} := I \times \cdots \times \underbrace{1}_{i^{th}\text{-place}} \times \cdots \times \underbrace{1}_{j^{th}\text{-place}} \times \cdots \times I$$

$$\mathbb{E}^{-1}_{ij} := I \times \cdots \times \underbrace{0}_{i^{th}\text{-place}} \times \cdots \times \underbrace{0}_{j^{th}\text{-place}} \times \cdots \times I$$

Proof. We know that $(\mathbb{R}^d)^* = \mathbb{R}^d$. Let $v^0_{ij} = (0, \ldots, \underbrace{-1}_{i\text{th-place}}, \ldots, \underbrace{1}_{j\text{th-place}}, \ldots, 0)$ and consider the affine hyperplane defined by

$$\mathbb{P}^0_{ij} = \{\mathbf{x} \in \mathbb{R}^d :< v^0_{ij}, \mathbf{x} >= 1\}$$

Then we have that for all $\mathbf{x} \in C^d$ we have that

$$< \mathbf{x}, v^0_{ij} >= x_i - x_j \leq 1$$

and $C^d \cap \mathbb{P}^0_{ij}=$

$$= \{\mathbf{x} \in C^d :< v^0_{ij}, \mathbf{x} >= 1\}$$

$$= \{\mathbf{x} \in C^d : x_i - x_j = 1\}$$

$$= \mathbb{E}^0_{ij}$$

Similarly, we let $v^1_{ij} = (0, \ldots, \underbrace{1}_{i\text{th-place}}, \ldots, \underbrace{1}_{i\text{th-place}}, \ldots, 0)$ and consider the affine hyperplane defined by

$$\mathbb{P}^1_{ij} = \{\mathbf{x} \in \mathbb{R}^d :< v^1_{ij}, \mathbf{x} >= 2\}$$

Then we have that for all $\mathbf{x} \in C^d$ we have

$$< \mathbf{x}, v^1_{ij} >= x_i + x_j \leq 2$$

and $C^d \cap \mathbb{P}^1_{ij}=$

$$= \{\mathbf{x} \in C^d :< v^1_{ij}, \mathbf{x} >= 2\}$$

$$= \{\mathbf{x} \in C^d : x_i + x_j = 2\}$$

$$= \mathbb{E}^1_{ij}$$

Finally, we let $v^{-1}_{ij} = (0, \ldots, \underbrace{-1}_{_i\text{th-place}}, \ldots, \underbrace{-1}_{_i\text{th-place}}, \ldots, 0)$ and consider the affine hyperplane defined by

$$\mathbb{P}^{-1}_{ij} = \{\mathbf{x} \in \mathbb{R}^d :< v^{-1}_{ij}, \mathbf{x} >= 0\}$$

Then we have that for all $\mathbf{x} \in C^d$ we have

$$< \mathbf{x}, v^{-1}_{ij} >= -x_i - x_j \leq 0$$

and $C^d \cap \mathbb{P}^{-1}_{ij}=$

$$= \{\mathbf{x} \in C^d :< v^{-1}_{ij}, \mathbf{x} >= 0\}$$

$$= \{\mathbf{x} \in C^d : x_i + x_j = 0\}$$

$$= \mathbb{E}^{-1}_{ij}$$

Notice also that $\dim(\mathbb{E}^k_{ij}) = d - 2$ for all appropriate i, j, k and so $\mathbb{E}^k_{ij} \subset \sigma_{d-2}(C^d)$ and so

$$(\cup_{i \neq j} \mathbb{E}^0_{ij}) \cup (\cup_{i<j} \mathbb{E}^1_{ij}) \cup (\cup_{i<j} \mathbb{E}^{-1}_{ij}) \subset \sigma_{d-2}(C^d).$$

Again, by Remark 1.48, we know that $\sigma_{d-2}(C^d)$ must be contained in the union of the relative boundaries over all the faces in $\sigma_{d-1}(C^d)$ and $\sigma_d(C^d)$. But since $\sigma_{d-1}(C^d) = \mathrm{rb}(C^d) = \partial(C^d)$, we get that

$$\sigma_{d-2}(C^d) \subset \cup_{F \in \sigma_{d-1}(C^d)} \mathrm{rb}(F) = (\cup_{i \neq j} \mathbb{E}^0_{ij}) \cup (\cup_{i<j} \mathbb{E}^1_{ij}) \cup (\cup_{i<j} \mathbb{E}^{-1}_{ij})$$

and so we conclude that

$$\sigma_{d-2}(C^d) = (\cup_{i \neq j} \mathbb{E}^0_{ij}) \cup (\cup_{i<j} \mathbb{E}^1_{ij}) \cup (\cup_{i<j} \mathbb{E}^{-1}_{ij})$$

□

Theorem 2.3. $\zeta_{d-2}(C^d) = 2d(d-1)$.

Proof. We get

$$\begin{aligned}
\zeta_{d-2}(C^d) &= \mathcal{H}^{d-2}((\cup_{i \neq j} \mathbb{E}^0_{ij}) \cup (\cup_{i<j} \mathbb{E}^1_{ij}) \cup (\cup_{i<j} \mathbb{E}^{-1}_{ij})) \\
&= \textstyle\sum_{i \neq j} \mathcal{H}^{d-2}(\mathbb{E}^0_{ij}) + \sum_{i<j} \mathcal{H}^{d-2}(\mathbb{E}^1_{ij}) + \sum_{i<j} \mathcal{H}^{d-2}(\mathbb{E}^{-1}_{ij}) \\
&= \textstyle\sum_{i \neq j} 1 + \sum_{i<j} 2 \\
&= d(d-1) + 2\tfrac{d(d-1)}{2} \\
&= 2d(d-1).
\end{aligned}$$

Where the second equality is a consequence of the fact that $\mathcal{H}^{d-2}(\mathbb{E}^k_{ij} \cap \mathbb{E}^n_{lm}) = 0$ if $\mathbb{E}^k_{ij} \neq \mathbb{E}^n_{lm}$. □

2.2 Bounding balls for polytopes

To keep our minimizing sequence of polytopes from collapsing into a smaller dimensional convex set, we would like to show that there exists a ball that is contained inside each polytope whose radius is bounded from below by a constant that bounds the total edge length. We will also show that each polytope is enclosed in a larger concentric ball with radius determined by the radius of the smaller ball, which again is determined by the upper bound for the edge length.

Theorem 2.4. *Let d be a positive integer and let M=$M(d)$ be a positive constant that depends only on d. Then there exists a positive constant $r(d)$ such that for all d-polytopes K such that $\mathcal{H}^d(K) = 1$ and $\mathcal{H}^{d-2}(K) \leq M$, there exists a point $x \in K$ such that $B_{r(d)}(x) \subset K$.*

Proof. We proceed similarly to the proof found in [Ber02]. Let $r > 0$ be such that for all $x \in K$ we have $d(x, \partial P) \leq r$. Let $\{F_i = K \cap P_i\}$ be the set of facets (codimension one faces) of K. Let v_i be the orienting vector for P_i and consider $N_i = \{x + tv_i : x \in f_i, 0 \leq t \leq r\}$. Then, by assumption, we get that $P \subset \cup_i N_i$ which implies that

$$1 = \mathcal{H}^d(K) \leq \mathcal{H}^d(\cup_i N_i) \leq \sum_i \mathcal{H}^d(N_i) = \sum_i r(\mathcal{H}^{d-1}(F_i)) = r\zeta_{d-1}(K).$$

From [EGK64] there exists a constant $\gamma_d = \gamma(d, d-2, d-1)$ such that

$$\frac{\zeta_{d-1}(K)^{\frac{1}{d-1}}}{\zeta_{d-2}(K)^{\frac{1}{d-2}}} \leq \gamma_d.$$

Then we get

$$1 \leq r\zeta_{d-1}(K) \leq r\zeta_{d-2}(K)^{\frac{d-1}{d-2}}\gamma_d \leq rM^{\frac{d-1}{d-2}}\gamma_d = rc_d$$

or $r \geq \frac{1}{c_d}$ where c_d is a constant that depends only on d. Then we may pick $r(d)$ such that $0 < r(d) < \frac{1}{c_d}$, or, more specifically, let

$$r(d) = \frac{1}{2c_d} = \frac{1}{2M^{\frac{d-1}{d-2}}\gamma_d}.$$

$\square$

Remark 2.5. Note that since $\mathcal{H}^d(K) = 1$ and $B_{r(d)} \subset K$, then we necessarily get that

$$r(d) \leq \alpha(d)^{-\frac{1}{d}} \leq 1$$

where $\alpha(d) = \mathcal{H}^d(B_1)$.

Theorem 2.6. *Let K be a d-polytope such that $\mathcal{H}^d(K) = 1$ and $B_{r(d)}(x) \subset K$. Then there exists constant $R(d)$ such that $K \subset B_{R(d)}(x)$.*

Proof. By translating K, we may assume $x = 0$. Let $y \in K$ and let $P_y = \{v \in \mathbb{R}^d :< v, y >= 0\}$ be the plane oriented by y. Then consider the cone

$$S = \delta_y \times\!\!\!\times (P_y \cap B_{r(d)}(0)) := \{ty + (1-t)z : 0 \leq t \leq 1, z \in P_y \cap B_{r(d)}(0)\}.$$

By convexity of K, we know that $S \subset K$ and hence

$$\begin{aligned} 1 &= \mathcal{H}^d(K) \\ &\geq \mathcal{H}^d(S) \\ &= \int_0^{|y|} \pi\left(\frac{r(d)t}{|y|}\right)^2 dt \\ &= \frac{\pi r(d)^2 |y|}{3}. \end{aligned}$$

Therefore, $|y| \leq \frac{3}{\pi r(d)^2}$ and we let $R(d) = \frac{3}{\pi r(d)^2}$. $\square$

2.3 Minimizing sequence of polytopes

We provide some preliminary definitions and theorems concerning the Hausdorff metric.

2.3.1 Hausdorff metric

Let (X, ρ) be a metric space and $\mathcal{T}$ be the set of all non-empty, bounded closed subsets of X. Let $A, B \in \mathcal{T}$. Then define the *Hausdorff metric* as

$$\rho_H(A, B) := \max\{\sup_{x\in A} \inf_{y\in B} \rho(x, y), \sup_{y\in B} \inf_{x\in A} \rho(x, y)\}.$$

Because the sets are bounded, the distance between two sets or "points" in $\mathcal{T}$ must be finite. The axioms for a metric can be checked, and we state the following well-known fact:

Fact 2.7. *$(\mathcal{T}, \rho_H)$ is a metric space.*

Now let $K \subset \mathbb{R}^d$ be a compact set and let $\mathcal{K}$ be the set of closed, bounded, convex subsets of K. Then the Hausdorff metric is again a metric on $\mathcal{K}$ and we have the following theorem for the space itself:

Theorem 2.8 ([KW79]). *$(\mathcal{K}, d_H)$ is compact.*

Since $(\mathcal{K}, d_H)$ is a metric space, compactness is equivalent to sequential compactness, and hence $(\mathcal{K}, d_H)$ is sequentially compact.

Finally, we consider $\mathcal{H}^d : \mathcal{K} \to \mathbb{R}$ as a function from our space of closed, convex sets into the real numbers. We have the following useful theorem:

Theorem 2.9 ([Egg58]). *$\mathcal{H}^d : \mathcal{K} \to \mathbb{R}$ is continuous with respect to the Hausdorff metric.*

We note the restriction for the domain, i.e. the sets must be convex for this volume function to be continuous.

2.3.2 Properties of the convex limiting set

We now define $\mathcal{P} := \{\text{convex polytopes } T \subset \mathbb{R}^d : \mathcal{H}^d(T) = 1\}$ be the unit volume polytopes in $\mathbb{R}^d$. A direct consequence of the previous section is that $\mathcal{P}$ is non-empty because $C^d \in \mathcal{P}$. Define the constant

$$M = M(d) := \zeta_{d-2}(C^d) = 2d(d-1).$$

Again, since $\mathcal{P}$ is nonempty, we may define

$$m := \inf_{T \in \mathcal{P}} \zeta_{d-2}(P).$$

We know from Theorem 1.13 that

$$m = \frac{1}{\gamma(d, d-2, d)^{d-2}} > 0.$$

Also, since $C^d \in \mathcal{P}$, we get that $m \leq M$. Now let $\{P_i\}_{i=1}^{\infty} \subset \mathcal{P}$ such that

$$\lim_{i \to \infty} \zeta_{d-2}(P_i) = m.$$

We may assume, by taking a subsequence and relabeling, that $\zeta_{d-2}(P_i) \leq M$ for all i. By Theorem 2.4, we know that for each i, there exists $\mathbf{x}_i \in \mathbb{R}^d$ and a ball $B_{r(d)}(\mathbf{x}_i)$ such that $B_{r(d)}(\mathbf{x}_i) \subset P_i$. Since Hausdorff measure and convexity are invariant under translations, we may again assume that $\mathbf{x}_i = 0$ for all i. Then, by Theorem 2.6, we know that $P_i \subset B_{R(d)}(0)$ for all i and hence $P_i \subset \bar{B}_{R(d)}$. By Theorem 2.8, there exists a subsequence P_{i_k} and a closed, bounded, convex subset $P \subset \bar{B}_{R(d)}$ such that

$\lim_{k\to\infty} d_H(P_{i_k}, P) = 0$ and so $P_{i_k} \to P$ with respect to the Hausdorff metric. Again, relabel this subsequence as $\{P_i\}_{i=1}^{\infty}$.

Claim 2.10. $B_{r(d)} \subset P \subset \bar{B}_{R(d)}$

Proof. Suppose there exists a point $x \in B_{r(d)}$ such that $x \notin P$. Since $B_{r(d)} \subset P_i$ for all i, we know that $x \in P_i$ for all i. Suppose that $||x|| = r(d) - \epsilon$. Then $d(x, P_i) \geq \epsilon$ for all i and so $d_H(P, P_i) \geq \epsilon$. This is a contradiction, and hence $B_{r(d)} \subset P$. □

Because P_i and P are closed, we get a stronger statement:

Claim 2.11. *$\bar{B}_{r(d)} \subset P_i$ for all i and $\bar{B}_{r(d)} \subset P$*

Proof. Since P_i and P are closed, then $P_i \cap P$ is also closed. We therefore get

$$B_{r(d)} \subset P_i \cap P$$

which implies

$$\bar{B}_{r(d)} \subset \overline{P_i \cap P} = P_i \cap P.$$

□

To be able to compare edge lengths, we also want the limiting convex set to satisfy $\mathcal{H}^d(P) = 1$. Thus we have the following corollary:

Corollary 2.12. $\mathcal{H}^d(P) = 1$.

Proof. By Theorem 2.9, we know that since $P_i \to P$, then

$$\mathcal{H}^d(P) = \lim_{i\to\infty} \mathcal{H}^d(P_i) = \lim_{i\to\infty} 1 = 1.$$

□

Now let $\mathcal{B} = \{K \in \mathcal{K} : \bar{B}_{r(d)} \subset K, K \subset \bar{B}_{R(d)}, \mathcal{H}^d(K) = 1\}$. Using the previous statements, we have shown:

Theorem 2.13. *$\mathcal{B}$ is compact.*

We would like to summarize the results in this section into one of the main theorems for the paper:

Theorem 2.14 (Main theorem 1). *Let*

$$\mathcal{P} = \{d\text{-polytopes } T \subset \mathbb{R}^d : \mathcal{H}^d(T) = 1\}.$$

Then there exists a sequence $\{P_i\}_{i=1}^{\infty} \subset \mathcal{P}$ converging to a convex set $P \subset \mathbb{R}^d$, and constants $0 < r(d) < R(d)$ such that

(i). $\bar{B}_{r(d)} \subset P_i, P \subset B_{R(d)}$

(ii). $\mathcal{H}^d(P) = 1$

(iii). $\lim_{i\to\infty} \zeta_{d-2}(P_i) = \inf_{T\in\mathcal{P}} \zeta_{d-2}(T)$.

2.3.3 Boundary convergence

Since d-polytopes are closed, bounded, and convex, we get that the sequence of boundaries $\{\partial P_i\}_{i=1}^{\infty}$ of the converging sequence $\{P_i\}_{i=1}^{\infty}$ is also converging, and that $\lim_{i\to\infty} \partial P_i = \partial P$. We get this from the following theorem:

Theorem 2.15 ([Ber02]). *Let $A, B \subset \mathbb{R}^d$ be compact and convex. Then*

$$d_H(A, B) = d_H(\partial A, \partial B).$$

Therefore, since $d_H(P_i, P) \to 0$, we also know that $d_H(\partial P_i, \partial P) \to 0$ and so $\partial P_i \to \partial P$ in the Hausdorff metric:

Lemma 2.16. *$\partial P_i \to \partial P$.*

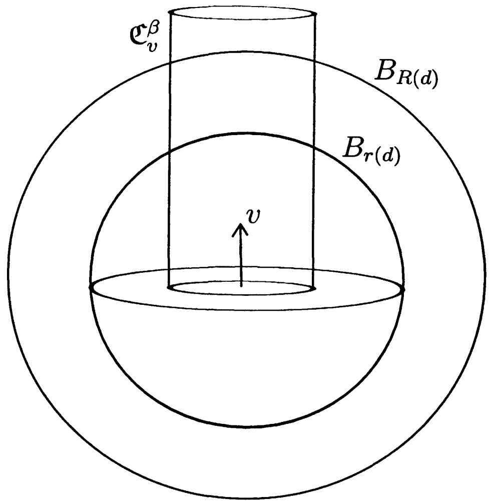

Figure 2.1: Region cut out by $\bar{B}_{R(d)} \setminus B_{r(d)}$ intersected with the half cylinder $\mathfrak{C}_v^\beta$.

2.4 Cylindrical partition

We now want to partition the spherical region $\bar{B}_R(d) \setminus B_{r(d)}$ into finitely many regions so we may view the boundaries of the polytopes as graphs of Lipschitz functions over a bounded region in a hyperplane. Let $0 < \beta < 1$. For $v \in S^d$, let H_v be the halfspace defined by

$$H_v := \{x \in \mathbb{R}^d :< x, v >\geq 0\}$$

and define the half-cylinder $\mathfrak{C}_v^\beta$ to be the cylinder with axis v and radius $\beta r(d)$ intersected with H_v:

$$\mathfrak{C}_v^\beta := \{x \in \mathbb{R}^d : ||x- < x, v > v|| < \beta r(d)\} \cap H_v.$$

2.4.1 Finite Open Covering

Now, for each $v \in S^d$, let $U_v := \mathfrak{C}_v^{\frac{1}{2}} \cap H_v \cap (\bar{B}_{R(d)} \setminus B_{r(d)})$. Then $\{U_v\}_{v \in S^d}$ is an open covering of $(\bar{B}_{R(d)} \setminus B_{r(d)})$, which is compact. Therefore, there exists a finite covering, say $\{U_{v_n}\}_{n=1}^N$.

2.5 Boundary functions

We now wish to consider the set $G_i^n := \partial P_i \cap \mathfrak{C}_{v_n}^\beta$ as the graph of a function f_i^n over the region $R_{v_n}^\beta := Q_{v_n} \cap \mathfrak{C}_{v_n}^\beta$ where

$$Q_{v_n} := \{x \in \mathbb{R}^d :< x, v_n >= 0\}$$

is the hyperplane defined by v_n. We wish to show that G_i^n can actually be considered as the graph of a function. For simplicity of notation, we may assume, after rotation, that $v_n = e_d$. We consider the object

$$T_i = P_i \cap \mathfrak{C}_{e_d}^\beta \cap H_{e_d},$$

the intersection of a convex polytope with the half cylinder. Each of these objects is a convex set, and hence T_i must be convex. We then let $p \in R_{e_d}$ and consider the affine vertical line through p, or $l = \{p + te_d : t \in \mathbb{R}\}$. This is again a convex set and so $T_i \cap l$ must also be convex. This is clearly a one-dimensional convex set and so must be a closed line segment. We know by construction that $p \in T_i$ is one endpoint of this segment because $T_i \subset H_{e_d}$. We denote by $p + ce_d$ the other endpoint of the line segment and let $f_i : R_{e_d} \to \mathbb{R}$ be defined by $f(p) = c$ where $c = \max\{t : p + te_d \in T_i\}$. By convexity and closedness, we know that $p + f(p)e_d \in \partial P_i$ and hence $p + f(p)e_d \in G_i$. Hence, we have shown the following:

Lemma 2.17. *There exists functions $f_i^n : R_{v_n} \to \mathbb{R}$ such that graph(f_i^n) $= G_i^n$. Similarly, there exists functions $f^n : R_{v_n} \to \mathbb{R}$ such that graph(f^n) $= G^n$.*

2.5.1 Lipschitz estimates and various convergences

So, if we fix our region by fixing v_n, we have a sequence of functions lying over $R^{\beta}_{v_n}$, each corresponding to the sequence of polytopes P_i and P. It is also convenient to consider the projection map $\Pi_{v_n} : \mathbb{R}^d \to H_{v_n}$ defined by

$$\Pi_{v_n}(p) = p - < p, v_n > v_n.$$

We wish to prove some properties of the functions f_i^n and f^n:

Theorem 2.18. *The functions $f_i^n, f^n : R^{\beta}_{v_n} \to \mathbb{R}$ are Lipschitz with Lipschitz constant*

$$Lip(f_i^n) \leq C_\beta(d) := \frac{R(d)}{r(d)\sqrt{(1-\beta^2)}}.$$

Proof. Again, without loss of generality, we assume that $v_n = e_d$ and, for ease of notation, let $f_i^n = h$. Let $p, p' \in R_{e_d}$. Without loss of generality, assume that $h(p') \leq h(p)$. Let W be the 2-plane defined by $p + h(p)e_d$, p, and p'. By rotational symmetry, we may assume this 2-plane is spanned by the x_1 and x_d coordinates where $x_2^2 + \ldots + x_{d-1}^2 = \alpha^2 < (\beta r(d))^2$. Then we know that

$$B_{R(d)} \cap W = \{(x_1, x_d) \in W : x_1^2 + x_d^2 < R(d)^2 - \alpha^2\}$$

and

$$B_{r(d)} \cap W = \{(x_1, x_2) \in W : x_1^2 + x_d^2 < r(d)^2 - \alpha^2\}$$

and similarly for the half-cylinder:

$$\mathfrak{C}^{\beta}_{e_d} \cap W = \{(x_1, x_2) \in W : x_1^2 < (\beta r(d))^2 - \alpha^2, x^d \geq 0\}.$$

Let $B = H_{e_d} \cap B_{r(d)}$ and consider the cone $CN = \delta_{(p+h(p)e_d)} \times\!\!\times (S_{r(d)} \cap H_{e_d})$. By convexity, we know that $CN \subset P_i$ and has slope

$$m \leq C_\beta(d) = \frac{R(d)}{r(d)\sqrt{(1-\beta^2)}}.$$

Figure 2.2 shows the intersection $CN \cap W$. Therefore, we know that if

$$l = h(p) - ||p - p'||m,$$

then $p' + le_d \in CN \subset P_i$. By definition of h, we get that

$$h(p') \geq h(p) - ||p - p'||m$$

which implies

$$h(p) - h(p') \leq m||p - p'|| \leq C_\beta(d)||p - p'||.$$

Because we assumed that $h(p') \leq h(p)$, we conclude

$$|h(p) - h(p')| \leq C_\beta(d)||p - p'||.$$

□

Consequently, we note the following corollary:

Corollary 2.19. *f_i^n, f^n are uniformly continuous on $R_{v_n}^\beta$ for all i, j and $0 < \beta < 1$.*

We also know that $G_i^n \to G^n$ in the sense of Hausdorff. Using this and the fact that the functions are Lipschitz, we can prove the following theorem about the convergence of f_i^n to f^n. To do this, we present the following lemma:

Lemma 2.20. *Let $s \in \partial P_i$ such that $\Pi_{e_d}(s) \in R_{e_d}^\gamma$. If $s \in H_{e_d}$, then $B(s, r(d)\sqrt{1-\gamma^2}) \subset H_{e_d}$.*

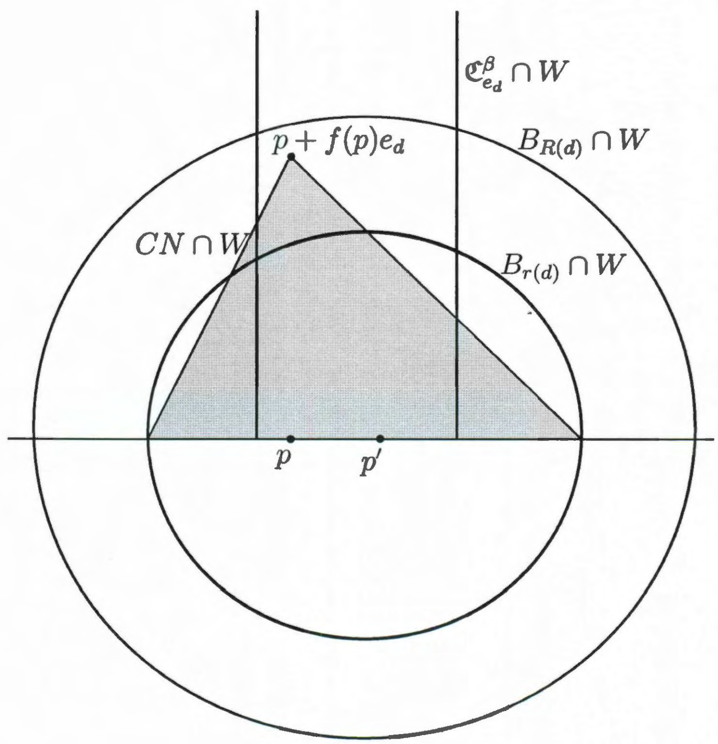

Figure 2.2: The Lipschitz constants for the functions f_i^n are bounded by the slope of the cone CN.

Proof. Since we know that $\bar{B}_r(d) \subset P_i$, then

$$||s||^2 = ||\Pi(s) + s_d e_d||^2 = ||\Pi(s)||^2 + s_d^2 \geq r(d)^2.$$

Since $\Pi(s) \in R^{\gamma}_{e_d}$, then we get that $s_d^2 \geq r_d^2 - (\gamma r_d)^2$, which implies $|s_d| \geq r(d)\sqrt{1-\gamma^2}$. Therefore, $d(s, Q_{e_d}) = |s_d| \geq r(d)\sqrt{1-\gamma^2}$, and so

$$B(s, r(d)\sqrt{1-\gamma^2}) \subset H_{e_d}.$$

□

Using the previous lemma, we arrive at the following theorem:

Theorem 2.21. *$f_i^n \rightarrow_{i\rightarrow\infty} f^n$ in $L^\infty(R^{\beta}_{v_n})$ for $0 < \beta < 1$.*

Proof. We again assume that $v_n = e_d$ and let $f_i^n = f_i$ and $f^n = f$. Let $\epsilon > 0$, let $\gamma = \frac{1+\beta}{2} < 1$ and let

$$\delta = \min\{\frac{\epsilon}{C_\gamma(d) + 2}, (\gamma - \beta) r(d)\}.$$

Let $N \in \mathbb{N}$ be such that $i > N$ implies $d_H(\partial P_i, \partial P) \leq \delta$. Let $p \in R^{\beta}_{e_d}$ and let $q_1 = p + f(p)e_d \in \partial P$. Then there exists $q_2 \in \partial P_i$ such that $d(q_1, q_2) \leq \delta$. Then we also know by projection that

$$d(\Pi(q_1), \Pi(q_2)) = d(p, \Pi(q_2)) \leq d(q_1, q_2) \leq \delta.$$

Let $\Pi(q_2) = p'$. Since $\delta \leq (\gamma - \beta) r(d)$, we know that $p' \in R^{\gamma}_{e_d}$. But we can have two possible points that project to p' from ∂P_i. Using the previous lemma, since

$$d(q_1, q_2) \leq \delta \leq r(d)(\gamma - \beta) \leq r(d)\sqrt{1-\gamma^2}$$

for our chosen value of γ, then we know that q_2 must be in H_{e_d} and hence $q_2 \in G_i$.

Now, let $q_3 = p + f_i(p)e_d$. Then we get the following:

$$\begin{aligned}
|f_i(p) - f(p)| &= d(q_1, q_3) \\
&\leq d(q_1, q_2) + d(q_2, q_3) \\
&\leq \delta + ||(p + f_i(p)e_d) - (p' + f_i(p')e_d)|| \\
&\leq \delta + ||p - p'|| + |f_i(p) - f_i(p')| \\
&\leq \delta + \delta + C_\gamma(d)||p - p'|| \\
&\leq \delta + \delta + C_\gamma(d)\delta \\
&= \delta(C_\gamma(d) + 2) \\
&\leq \epsilon.
\end{aligned}$$

□

Because $R^\beta_{v_n}$ has finite $\mathcal{H}^{d-1}$ measure, we get the following corollary:

Corollary 2.22. *$f^n_i \to f^n$ in $L^p(R^\beta_{v_n})$ for all $p \geq 1$, $0 < \beta < 1$, and $1 \leq n \leq N$.*

One more property we will need for the functions f^n_i and f^n is that they are concave. This is a direct consequence of the convexity of the polytopes P_i and P:

Lemma 2.23. *The functions f^n_i and f^n are concave.*

Proof. Again we may assume that $v_n = e_d$ and let $f^n = f$. Let $p, p' \in R^\beta_{e_d}$. Then we know that $p + f(p)e_d \in \partial P \subset P$ and similarly $p' + f(p')e_d \in \partial P \subset P$. By convexity of P, we get

$$\{[\lambda p + (1-\lambda)p'] + [\lambda f(p) + (1-\lambda)f(p')]e_d : \lambda \in (0,1)\} \subset P.$$

Therefore, by the definition of f, we get that

$$f(\lambda p + (1-\lambda)p') \geq \lambda f(p) + (1-\lambda)f(p').$$

The proof is similar for f^n_i and hence f^n_i and f^n are concave. □

2.5.2 Differentiability

For this section and for the rest of the paper (until otherwise noted), we will fix the radius of our cylinder $\mathfrak{C}^{\beta}_{v_n} = \mathfrak{C}^{\frac{1}{2}}_{v_n} = \mathfrak{C}_{v_n}$ and, after rotation, we will assume $\mathfrak{C}_{v_n} = \mathfrak{C}_{e_d}$. Suppressing further corresponding notation, we let $f_i^n = f_i$ be the functions corresponding to the graphs $G_i^n = G_i$. Similarly, we let $f^n = f$ correspond to $G^n = G$. Also, since $R^{\frac{1}{2}}_{v_n} = R_{e_d}$, we may view the functions $f_i : R_{e_d} \to \mathbb{R}$ as functions $f_i : B_s \subset \mathbb{R}^{d-1} \to \mathbb{R}$ where $s = r(d)/2$ or $\beta = \frac{1}{2}$, i.e. $f_i(x_1, \ldots, x_{d-1}, 0) = f_i(x_1, \ldots, x_{d-1})$. Similarly for f. Using the previous section, we know that

$$Lip(f_i), Lip(f) \leq C_{\frac{1}{2}}(d) =: C_1(d)$$

and

$$f_i \to f \text{ in } L^1(B_s).$$

We let $\Pi : \mathbb{R}^d \to Q_{e_d}$ be the standard projection onto the hyperplane defined by e_d. We recall that P_i are polytopes and Corollary 1.50, which says that $\sigma_h(P_i) \subset \sigma_k(P_i)$ if $h \leq k$. This, together with Remark 1.48, gives us a partition of the boundary of P_i as

$$\partial P_i = \Big(\bigcup_{F_k \in \sigma_{d-1}(P_i)} \mathrm{ri}(F_k) \Big) \bigcup \sigma_{d-2}(P_i).$$

We then define the set $E_i^n = E_i := G_i \cap \sigma_{d-2}(P_i)$ and think of this as the "edge set" of the convex polytope P_i that lies in the cylinder $\mathfrak{C}_{v_n}$ which, here, we have fixed as $\mathfrak{C}_{e_d}$. We also define the set $J_i^n = J_i := \Pi(E_i) \subset R_{e_d}$. We also note that

$$\mathcal{H}^{d-2}(J_i) = \mathcal{H}^{d-2}(\Pi(E_i)) \leq \mathcal{H}^{d-2}(E_i) \leq \mathcal{H}^{d-2}(\sigma_{d-2}(P_i)) \leq M.$$

In particular, $\mathcal{H}^{d-1}(J_i) = 0$.

Now consider a point $p \in R_{e_d} \setminus J_i$. Since $\Pi|_{G_i}$ is a homeomorphism, there exists a neighborhood of p where the graph of f_i is a relatively open subset of a hyperplane. Therefore, f_i must be differentiable at p, which implies that f_i is differentiable $\mathcal{H}^{d-1}$-a.e., or ∇f_i exists $\mathcal{H}^{d-1}$-a.e. (We also know that since f_i is Lipschitz, then it must be differentiable a.e.) Also, since the functions f_i are Lipschitz, we get that

$$|\nabla f_i| \leq C_1(d) \ \ \mathcal{H}^{d-1}\text{-a.e.}$$

Similarly, since f is Lipschitz, we get that f is differentiable $\mathcal{H}^{d-1}$-a.e in B_s. Let ∇f be this derivative and again note that since f is Lipschitz, we get

$$|\nabla f| \leq C_1(d) \ \ \mathcal{H}^{d-1}\text{-a.e.}$$

Henceforth, we define

$$g_i^j := (\nabla f_i)_j$$

and

$$\frac{\partial f}{\partial x_j} := (\nabla f)_j.$$

2.6 Properties of J_i

Because Π is Lipschitz ($\text{Lip}(\Pi) = 1$), we have a bilipschitz correlation between G_i and B_s and also between E_i and J_i. We wish to show some properties of J_i which result from the the properties of f_i and projections. We have some preliminary lemmas and facts:

Lemma 2.24. *Let $S \subset \mathbb{R}^d$ be convex and let $\Pi : \mathbb{R}^d \to Q$ be projection onto a hyperplane. Then $\Pi(S)$ is convex.*

Proof. By extending a basis from the vector that defines the hyperplane, we may assume that $Q = Q_{e_d}$. Let $p, q \in \Pi(S)$ where $p = (p_1, \ldots, p_{d-1}, 0)$ and $q = (q_1, \ldots, q_{d-1}, 0)$. Then there exists $p_d, q_d \in \mathbb{R}$ such that $(p_1, \ldots, p_d) \in S$ and $(q_1, \ldots, q_d) \in S$. Then, since S is convex, we know

$$\{\lambda(p_1, \ldots, p_d) + (1-\lambda)(q_1, \ldots, q_d) : \lambda \in (0,1)\} \subset S$$

which implies

$$\Pi(\{\lambda(p_1, \ldots, p_d) + (1-\lambda)(q_1, \ldots, q_d) : \lambda \in (0,1)\}) \subset \Pi(S)$$

and so we get

$$\{\lambda p + (1-\lambda)q : \lambda \in (0,1)\} \subset \Pi(S).$$

Therefore $\Pi(S)$ is convex. □

For a convex set, we have two different notions of dimension that we would like to show are equivalent.

Lemma 2.25. *Let $S \subset \mathbb{R}^d$ be convex. Then*

$$\dim(S) = \mathcal{H}_{\dim}(S)$$

Proof. Suppose $\dim S = s$. Then there exist $s+1$ non-coplanar points $t_1, \ldots, t_{s+1} \in S$. Since S is convex, this implies that $\mathrm{conv}(t_1, \ldots, t_{s+1}) = \eta \subset S$. But since $\eta \subset S$ is an s-simplex, we know that $s = \mathcal{H}_{\dim}(\sigma) \leq \mathcal{H}_{\dim}(S)$. Also, by definition, there exists an s-dimensional linear affine subspace L such that $\mathrm{aff}(S) = L$. In particular, $S \subset L$. Therefore, $\mathcal{H}_{\dim}(S) \leq \mathcal{H}_{\dim}(L) = s$, and so $\mathcal{H}_{\dim}(S) = \dim(S)$. □

We present the following commonly used fact about Hausdorff measure:

Fact 2.26 ([Fed69], §2.10.11). *Let $f : \mathbb{R}^N \to \mathbb{R}^m$ be a Lipschitz function and $E \subset \mathbb{R}^N$.*

Then

$$\mathcal{H}^k(f(E)) \leq [Lip(f)]^k \mathcal{H}^k(E).$$

Using this, we arrive at the following useful corollary:

Corollary 2.27. *Let $f : \mathbb{R}^N \to \mathbb{R}^m$ be bilipschitz and $E \subset \mathbb{R}^N$. Then*

$$\mathcal{H}_{\text{dim}}(E) = \mathcal{H}_{\text{dim}}(f(E)).$$

Proof. From the previous fact, we know that if $\mathcal{H}^k(E) = 0$, then $\mathcal{H}^k(f(E)) = 0$ and therefore, by definition, $\mathcal{H}_{\text{dim}}(f(E)) \leq \mathcal{H}_{\text{dim}}(E)$. Similarly, we use the fact again with f^{-1} and $E = f^{-1}(f(E))$ to get $\mathcal{H}_{\text{dim}}(E) \leq \mathcal{H}_{dim}(f(E))$. □

Therefore, since P_i has finitely many $(d-2)$-faces, this implies that there are only finitely many that can intersect the cylinder $\mathfrak{C}_{e_d}$. Since each of these faces are closed and convex and $\mathfrak{C}_{e_d}$ is open and convex, then the intersection must be a convex set with dimension $d-2$. Let $F_1, \ldots, F_m$ be these resulting intersections with finitely many $d-2$-faces. Then

$$J_i = \bigcup_{l=1}^{m} \Pi_{e_d}(F_l) = \Pi_{e_d}(E_i).$$

Since convexity and dimension are preserved under this correlation, then we arrive at the following conclusion:

Lemma 2.28. *J_i is the finite union of $(d-2)$-dimensional convex sets.*

Since each F_l was relatively closed in $\mathfrak{C}_{e_d}$, each $\Pi(F_l)$ is relatively closed in B_s and hence J_i is relatively closed in B_s. We show this fact directly using the fact that $\sigma_{d-2}(P_i)$ is closed:

Lemma 2.29. *J_i is relatively closed as a subset of B_s.*

Proof. Let $\{x^h\}$ be a sequence of points in J_i converging to a point $p \in B_s$. Let $\epsilon > 0$. Then there exists $N > 0$ such that $||x^k - x^l|| \leq \frac{\epsilon}{1+C_1(d)}$. Let $y^h = (x_1^h, \ldots, x_{d-1}^h, f_i(x^h))$

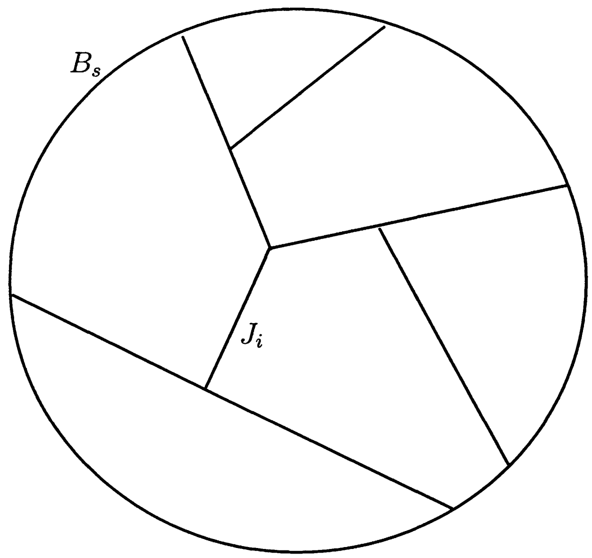

Figure 2.3: An example of a possible jump set J_i for the polyhedron P_i over B_s

and consider $\{y^h\} \subset E_i$. Let $k, l > 0$. Then

$$\begin{aligned} ||y^k - y^l|| &\leq ||x^k - x^l|| + |f_i(x^l) - f_i(x^h)| \\ &\leq ||x^k - x^l|| + Lip(f_i)||x^k - x^l|| \\ &\leq (1 + C_1(d))\tfrac{\epsilon}{1+C_1(d)} \\ &= \epsilon. \end{aligned}$$

Therefore, $\{y^h\}$ is converging to a point q, and since $\sigma_{d-2}(P_i)$ closed, we know that $q \in \sigma_{d-2}(P_i)$. We also know that $x^h = \Pi(y^h) \to \Pi(q)$. Then by uniqueness, $\Pi(q) = p$. Since $p \in B_s$, we know that $q \in E_i$ and hence $p \in J_i$. $\square$

2.7 BV boundedness

For any function $u \in BV(\Omega)$, we have the mutually singular decomposition of the associated Radon measure for Du:

$$Du = Du^a + Du^j + Du^c$$

where Du^a, Du^j, and Du^c are the absolutely continuous part, the jump part, and the cantor part of the measure Du, respectively. For definitions and a more detailed discussion, see [AFP00], p.140 and Corollary 3.33.

Definition 2.30. We say $u \in BV(\Omega)$ is in $SBV(\Omega)$ if $Du^c = 0$.

For a more thorough treatment of SBV, see [AFP00], §4.1. We will now show the functions g_i^j are in $SBV(B_s)$ and are uniformly bounded in the BV norm. We also have the following proposition:

Proposition 2.31 ([AFP00] §4.1). *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $K \subset \mathbb{R}^N$ closed and assume that $\mathcal{H}^{N-1}(K \cap \Omega) < \infty$. Then, any function $u : \Omega \to \mathbb{R}$ that belongs to $L^\infty(\Omega \setminus K) \cap W^{1,1}(\Omega \setminus K)$ belongs also to $SBV(\Omega)$ and satisfies $\mathcal{H}^{N-1}(S_u \setminus K) = 0$.*

We use the previous proposition to prove the following lemma:

Lemma 2.32. *$g_i^j \in SBV(B_s)$ and $S_{g_i^j} \subset J_i$.*

Proof. Recall $(\nabla f_i)_j = \frac{\partial f_i}{\partial x_j} = g_i^j$. By Lemma 2.29 we know that J_i is relatively closed in B_s. Then since $|\nabla f_i| \leq C_1(d)$, we get that $||g_i^j||_{L^\infty(B_s)} \leq C_1(d)$. Since f_i was linear almost everywhere (off of J_i), we know that g_i^j is locally constant in $B_s \setminus J_i$, and hence $\nabla g_i^j = 0$ on $B_s \setminus J_i$. In particular, $g_i^j \in C^\infty(B_s \setminus J_i)$. We know previously that $\mathcal{H}^{d-2}(J_i) \leq M$, and therefore, by Proposition 2.31, $g_i^j \in SBV(B_s)$. Also, since g_i^j is differentiable on $B_s \setminus J_i$, this implies further that $S_{g_i^j} \subset J_i$ by definition. $\square$

Remark 2.33. Note that $\mathcal{J}g_i^j \subset S_{g_i^j} \subset J_i$ for all $1 \leq j \leq (d-1)$.

Thus, we have a uniform bound on the BV norms of g_i^j:

Lemma 2.34. *There exists constant $C_2(d)$ such that*

$$||g_i^j||_{BV(B_s)} \leq C_2(d)$$

for all i and $1 \leq j \leq d-1$.

Proof. By Lemma 2.32 and Remark 2.33, we get:

$$\begin{aligned}
||g_i^j||_{BV(B_s)} &= ||g_i^j||_{L^1(B_s)} + ||Dg_i^j||(B_s) \\
&= \int_{B_s} |g_i^j| d\mathcal{H}^{d-1} + \int_{B_s} |\nabla g_i^j| d\mathcal{H}^{d-1} + \int_{\mathcal{J}g_i^j} [(g_i^j)^+ - (g_i^j)^-] \, d\mathcal{H}^{d-2} \\
&\leq C_1(d)\mathcal{H}^{d-1}(B_s) + \int_{J_i} 2C_1(d) \, d\mathcal{H}^{d-2} \\
&\leq C_1(d)\alpha(d-1)(r(d)/2)^{d-1} + 2C_1(d)\mathcal{H}^{d-2}(J_i) \\
&\leq C_1(d)\alpha(d-1)(r(d)/2)^{d-1} + 2C_1(d)M.
\end{aligned}$$

where $\alpha(d-1) = \mathcal{H}^{d-1}(B_1)$. The third inequality is obtained because $|g_i^j| \leq C_1$ $\mathcal{H}^{d-1}$-a.e. and by Remark 2.33. We let

$$C_2(d) := C_1(d)\alpha(d-1)(r(d)/2)^{d-1} + 2C_1(d)M$$

and note that M is also only dependent on d. $\square$

2.8 Weak* convergence in BV

We would now like to show that the partial derivatives of the boundary functions converge weakly* in $BV(B_s)$. We will make precise what this means, but first we present the following lemma:

Lemma 2.35. $g_i^j \to \frac{\partial f}{\partial x_j}$ *strongly in $L^1(B_s)$ for $1 \leq j \leq d-1$.*

Proof. By Theorem 2.33 and BV compactness, there exists subsequence $g^j_{i_l}$ and $h^j \in BV(B_s)$ such that $g^j_{i_l} \to h^j$ in $L^1(B_s)$. This implies that $g^j_{i_l} \rightharpoonup h^j$ weakly in $L^1(B_s)$. Let

$$\phi \in C^1_c(B_s) \subset L^\infty(B_s) = (L^1(B_s))^*.$$

We also note from previous statements that we have $f_i \to f$ in $L^1(B_s)$, and since the functions f_i and f are Lipschitz, then they are also absolutely continuous. We have:

$$\begin{aligned} \int_{B_s} h^j \phi \, d\mathcal{H}^{d-1} &= \lim_{l\to\infty} \int_{B_s} \frac{\partial f_{i_l}}{\partial x_j} \phi \, d\mathcal{H}^{d-1} \\ &= -\lim \int f_{i_l} \frac{\partial \phi}{\partial x_j} \, d\mathcal{H}^{d-1} \\ &= -\int f \frac{\partial \phi}{\partial x_j} \, d\mathcal{H}^{d-1} \\ &= \int_{B_s} \frac{\partial f}{\partial x_j} \phi \, d\mathcal{H}^{d-1} \end{aligned}$$

Therefore, $h^j = \frac{\partial f}{\partial x_j}$ $\mathcal{H}^{d-1}$-a.e. Since every subsequence must have a weakly convergent subsequence by BV compactness and this weak limit is independent of the subsequence, this implies that the original sequence g^j_i converges strongly to $(\nabla f)_j$ in $L^1(B_s)$. □

The structure of the dual space of the space of BV functions with the standard norm is not well known. Hence we will define what we mean for functions to converge weakly* in the space of BV functions:

Definition 2.36. Let $u, u_h \in [BV(\Omega)]^m$. We say that (u_h) *weakly* converges in* $[BV(\Omega)]^m$ *to* u if (u_h) converges to u in $[L^1(\Omega)]^m$ and Du_h weakly* converges to Du in Ω, i.e.

$$\lim_{h\to\infty} \int_\Omega \phi \, dDu_h = \int_\Omega \phi \, dDu \quad \text{for all } \phi \in C_0(\Omega).$$

The following proposition provides as equivalent characterization for weak* convergence in the space of BV functions:

Proposition 2.37 ([AFP00] §3.1). *Let* $(u_h) \subset [BV(\Omega)]^m$. *Then* (u_h) *weakly* con-*

verges to u in $[BV(\Omega)]^m$ if and only if (u_h) is bounded in $[BV(\Omega)]^m$ and converges to u in $[L^1(\Omega)]^m$.

Thus, we have the following lemma resulting from the previous proposition:

Theorem 2.38. *$\frac{\partial f}{\partial x_j} \in BV(B_s)$ and g_i^j weakly* converges in $BV(B_s)$ to $\frac{\partial f}{\partial x_j}$.*

Proof. From Lemma 2.35, we have $g_i^j \to \frac{\partial f}{\partial x_j}$ strongly in $L^1(B_s)$. We also know from Theorem 2.34 that $||g_i^j||_{BV(B_s)} \leq C_2(d)$ for all i. Applying Proposition 2.37, we arrive at our desired conclusion. □

A corollary of the proof of the previous theorem tells us the following:

Corollary 2.39. *$\frac{\partial f}{\partial x_j} \in BV(B_s)$.*

2.8.1 $\nabla f \in [SBV(B_s)]^{d-1}$

We would now like to show that $\nabla f \in [SBV(B_s)]^{d-1}$. From before, we know that off of J_i that g_i^j is locally constant. Since $\mathcal{H}^{d-1}(J_i) \leq M < \infty$, we get that $\nabla g_i^j = 0$ $\mathcal{H}^{d-1}$-a.e. in B_s. We now present the following theorem concerning the closure of SBV under certain constraints:

Theorem 2.40 ([AFP00] §4.1)**.** *Let $\varphi : [0, \infty) \to [0, \infty]$, $\theta : (0, \infty) \to (0, \infty]$ be lower semicontinuous increasing functions and assume that*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty, \ \lim_{t\to 0} \frac{\theta(t)}{t} = \infty.$$

Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and let $(u_h) \subset SBV(\Omega)$ such that

$$\sup_h \{ \int_\Omega \phi(|\nabla u_h|)\, dx + \int_{J_{u_h}} \theta(|u_h^+ - u_h^-|)\, d\mathcal{H}^{N-1} \} < \infty$$

If (u_h) weak converges in $BV(\Omega)$ to u, then $u \in SBV(\Omega)$, the approximate gradients ∇u_h weakly converge to ∇u in $[L^1(\Omega)]^N$, $D^k u_h$ weak* converge to $D^k u$ in Ω and*

$$\int_\Omega \varphi(|\nabla u|)\, dx \leq \liminf_{h\to\infty} \int_\Omega \varphi(|\nabla u_h|)\, dx \text{ (if } \varphi \text{ is convex)}$$

$$\int_{J_u} \theta(|u^+ - u^-|)\, d\mathcal{H}^{N-1} \leq \liminf_{h\to\infty} \int_{J_{u_h}} \theta(|u_h^+ - u_h^-|)\, d\mathcal{H}^{N-1}$$

if θ is concave.

Using this and previous results, we arrive at the following conclusions:

Theorem 2.41. *Let $1 \leq j \leq d-1$. Then*
(i) $\frac{\partial f}{\partial x_j} \in SBV(B_s)$.
(ii) ∇g_i^j weakly converge to $\nabla(\frac{\partial f}{\partial x_j})$ in $[L^1(B_s)]^{d-1}$.
(iii) $D^k g_i^j$ weakly converge to $D^k \frac{\partial f}{\partial x_j}$ in B_s.*
(iv) $\nabla \frac{\partial f}{\partial x_j} = 0$ $\mathcal{H}^{d-1}$-a.e.
(v) $\mathcal{H}^{d-2}(\mathcal{J}\frac{\partial f}{\partial x_j}) \leq \liminf_{i\to\infty} \mathcal{H}^{d-2}(\mathcal{J} g_i^j)$.

Proof. We apply Theorem 2.40 using $\varphi(t) = t^2$ and $\theta(t) = 1$ and note that φ is convex and θ is concave in this case. From Theorem 2.38 we know that g_i^j weakly* converges in $BV(B_s)$ to $\frac{\partial f}{\partial x_j}$. Because $\nabla g_i^j = 0$ for all i and $\mathcal{J} g_i^j \subset J_i$, we get that

$$\begin{aligned} \sup_i\{\textstyle\int_{B_s} |\nabla g_i^j|^2\, dx + \int_{\mathcal{J} g_i^j} d\mathcal{H}^{d-2}\} &= \sup_i\{\mathcal{H}^{d-2}(\mathcal{J} g_i^j)\} \\ &\leq \sup_i\{\mathcal{H}^{d-2}(J_i)\} \\ &\leq M. \end{aligned}$$

Thus, (i), (ii), and (iii) follow. We also have that

$$\int_{B_s} |\nabla \frac{\partial f}{\partial x_j}|^2\, dx \leq \liminf_{i\to\infty} \int_{B_s} |\nabla g_i^j|^2\, dx = 0$$

which implies (iv). Finally, using Theorem 2.40 again, we get

$$\mathcal{H}^{d-2}(\mathcal{J}\frac{\partial f}{\partial x_j}) = \int_{\mathcal{J}\frac{\partial f}{\partial x_j}} d\mathcal{H}^{d-2} \leq \liminf_{i\to\infty} \int_{\mathcal{J}g_i^j} d\mathcal{H}^{d-2} = \liminf_{i\to\infty} \mathcal{H}^{d-2}(\mathcal{J}g_i^j)$$

which gives us (v). □

2.9 Gradients as piecewise constant functions

We would like to prove further regularity results about the functions $\frac{\partial f}{\partial x_j}$, more specifically that they are BV weakly* equivalent to Caccioppoli piecewise constant functions. To do this, we introduce some definitions and lemmas.

Lemma 2.42. *Let μ be a finite measure on Ω, $\{u_h\} \in L^\infty(\Omega)$, and assume there exists constant M such that $||u_h||_\infty \leq M$. Then u_h converges to v in measure if and only if $v \in L^1(\Omega)$ and $||u_h - v||_{L^1} \to 0$.*

Proof. First suppose that $v \in L^1(\Omega)$ and $||u_h - v||_1 \to 0$. Let $\epsilon > 0$ and let $A_h^\epsilon = \{|u_h - v| \geq \epsilon\}$. Then

$$\begin{aligned} \mu\{A_h^\epsilon\} &= \int_{A_h^\epsilon} d\mu \\ &\leq \int_{A_h^\epsilon} \frac{|u_h - v|}{\epsilon}\, d\mu \\ &\leq \frac{1}{\epsilon}\int_\Omega |u_h - v|\, d\mu \\ &= \frac{1}{\epsilon}||u_h - v||_1. \end{aligned}$$

Taking $h \to 0$, we get the first implication. Now suppose that u_h converges to v in measure. Then there exists a subsequence $\{u_{h_l}\}$ that converges to v μ-a.e. Since $||u_h||_\infty \leq M$, this implies also that $||v||_\infty \leq M$. Since μ is a finite measure, we know that $v \in L_1(\Omega)$. Let $\epsilon > 0$ and let $\delta = \frac{\epsilon}{2M+\mu(\Omega)}$. There exists $N > 0$ such that $h > N$

implies $\mu(A_h^\delta) \leq \delta$. Let $h > N$. Then we get that

$$\begin{aligned}
||u_h - v||_1 &= \int_\Omega |u_h - v| \, d\mu \\
&= \int_{A_h^\delta} |u_h - v| \, d\mu + \int_{\Omega \setminus A_h^\delta} |u_h - v| \, d\mu \\
&\leq 2M\mu(A_h^\delta) + \int_{\Omega \setminus A_h^\delta} \delta \, d\mu \\
&\leq \delta(2M + \mu(\Omega)) \\
&= \epsilon.
\end{aligned}$$

□

2.9.1 Caccioppoli partitions

We now want to define a piecewise constant function that also has bounded variation. To do this, we will introduce the concept of Caccioppoli partitions.

Definition 2.43. Let $\Omega \subset \mathbb{R}^N$ be an open set and $I \subset \mathbb{N}$; we say that a partition $\{D_i\}_{i\in I}$ of Ω is a *Caccioppoli partition if* $\Sigma_{i\in I} P(D_i, \Omega) < \infty$.

We state the following theorem concerning the compactness of Caccioppoli partitions which will be used later in the paper but is also used implicitly in the proof of the compactness of piecewise constant functions below.

Theorem 2.44 ([AFP00] §4.4). *Let $\{E_{i,h}\}_{i\in I}$, $h \in \mathbb{N}$, be Caccioppoli partitions of $\mathbb{R}^N$ satisfying*

$$L := \sup\{\sum_{i\in I} P(E_{i,h}, \mathbb{R}^N) : h \in \mathbb{N}\} < \infty.$$

Then, if either I is finite or the partitions are individually ordered by the volumes of the members, there exists a Caccioppoli partition $\{E_i\}_{i\in I}$ and a subsequence $(h(k))$ such that $(E_{i,h(k)})$ locally converges in measure in $\mathbb{R}^N$ to E_i for any $i \in I$.

Using this, we can then intuitively define a piecewise step function:

Definition 2.45. We say that $u : \Omega \to \mathbb{R}^m$ is *piecewise constant in* Ω if there exist a Caccioppoli partition and a map $t : I \to \mathbb{R}^m$ such that

$$u = \sum_{i \in I} t_i \chi_{D_i}.$$

Since G_i has only finitely many faces, there must be only finitely many connected regions in Ω. Since f_i is linear in each of these regions, g_i^j must be constant on each of the components. Let $D_{i,k}^j$ be such that

$$g_i^j = \sum_{k=1}^{N_i^j} t_{i,j}^k \chi_{D_{i,k}^j}$$

where N_i^j must be finite. By Theorem 1.51, we know that for $\mathcal{H}^{d-2}$-almost all points in J_i are contained in the reduced boundary of exactly two regions. Hence, we know that

$$\sum_{k=1}^{N_i^j} P(D_{i,k}^j, B_s) \leq 2\mathcal{H}^{d-2}(J_i) \leq 2M.$$

Therefore, g_i^j are piecewise constant functions for $i > 0$ and $1 \leq j \leq d-1$. We also have compactness in the space of piecewise constant functions under certain assumptions:

Theorem 2.46 ([AFP00] §4.4). *Let Ω be a bounded open set with Lipschitz boundary. Let $(u_h) \subset [SBV(\Omega)]^m$ be a sequence of piecewise constant functions such that $(||u_h||_\infty + \mathcal{H}^{N-1}(S_{u_h}))$ is bounded. Then, there exists a subsequence $(u_{h(k)})$ converging in measure to a piecewise constant function u.*

We note that this compactness gets us a limit only that converges in measure. But, combined with the previous lemma, we obtain the following theorem:

Theorem 2.47. *$\frac{\partial f}{\partial x_j}$ is equivalent to a piecewise constant function under the BV weak* topology.*

Proof. From the previous discussion, we know that g_i^j are piecewise constant functions. We also know that for all $i > 0$ that

$$||g_i^j||_\infty + \mathcal{H}^{d-2}(S_{g_i^j}) \leq C_1(d) + M.$$

By the compactness theorem, there exists a subsequence $(g_{i_h}^j)$ that converges in measure to a piecewise constant function w^j. Since this subsequence was uniformly bounded $(||g_{i_h}^j||_\infty \leq C_1(d))$, Lemma 2.42 implies that $g_{i_h}^j$ converges to w^j strongly in $L^1(B_s)$. We already know that this subsequence converges to $\frac{\partial f}{\partial x_j}$ in $L^1(B_s)$ by Lemma 2.35, this implies that $w^j = \frac{\partial f}{\partial x_j}$ $\mathcal{H}^{d-1}$-a.e. Therefore, since every subsequence has a subsequence that converges to the same limit, we get that the original sequence has the same limit, so that $g_i^j \to w^j$ in $L^1(B_s)$. Again, by Theorem 2.37, we get that $g_i^j \to w^j$ weakly* in $BV(B_s)$, or

$$\lim_{i\to\infty} \int_{B_s} \phi \; dDg_i^j = \int_{B_s} \phi \; dDw^j$$

for all $\phi \in C_0(B_s)$. Therefore,

$$\int_{B_s} \phi \; dD\frac{\partial f}{\partial x_j} = \int_{B_s} \phi \; dDw^j$$

for all $\phi \in C_0(B_s)$. $\square$

Chapter 3

Boundary Regularity

From Chapter 2, we have a limiting convex set P whose boundary is locally the graph of a Lipschitz function. The gradient of this function is in $[SBV]^{d-1}$. In this chapter we show there exists a lower bound on the density of the discontinuity set for the gradient of this boundary function. We show the image of the points in the complement of the closure of the discontinuity set must be "face points", and the complement of this set in the boundary of P (the generalized edge set) must have finite $\mathcal{H}^{d-2}$ measure. Finally, we show that $\mathcal{H}^{d-2}$ almost all the points in ∂P must be either "face points" or "good edge points". These are Main Theorem 2 and Main Theorem 3 and are found in §3.2 and §3.3, respectively.

3.1 Density lower bound

In this section, we will show that the density for the discontinuity set of ∇f is bounded from below.

Up to this point, we have shown for all j, that $\frac{\partial f}{\partial x_j} \in SBV(B_s)$ and

$$\mathcal{H}^{d-2}(S_{\frac{\partial f}{\partial x_j}}) = \mathcal{H}^{d-2}(\mathcal{J}\frac{\partial f}{\partial x_j}) \leq M.$$

Now let

$$S := \bigcup_{j=1}^{d-1} S_{\frac{\partial f}{\partial x_j}}.$$

Thus we get that $\mathcal{H}^{d-2}(S) \leq (d-1)M$. We present the following theorem which is equivalent to a constancy theorem:

Theorem 3.1 ([AFP00] §3.1). *Let $u \in BV_{loc}(\Omega)$. If $Du = 0$, then u is constant in any connected component of Ω.*

From the compactness theorem, we know that $\nabla \frac{\partial f}{\partial x_j} = 0$ for all j, and therefore $D\frac{\partial f}{\partial x_j} = 0$ on $B_s \backslash S_{\frac{\partial f}{\partial x_j}}$. Therefore $\frac{\partial f}{\partial x_j}$ is constant on connected components of $B_s \backslash S_{\frac{\partial f}{\partial x_j}}$ and this implies that ∇f is constant on connected components of $B_s \setminus S$. From the continuity and convexity of f, we get the following Lemma 3.3, but first we prove a preliminary lemma.

Lemma 3.2. *Let P be a closed convex set that contains the ball $\bar{B}_\rho$. Let HP be a supporting hyperplane for P. Then*

$$HP \cap B_\rho = \emptyset.$$

Proof. Suppose not. Then there necessarily exists points in the ball on either side of the hyperplane HP. Since P must lie completely on one side or the other of the hyperplane, then P cannot contain the ball $\bar{B}_\rho$, which is a contradiction. □

Lemma 3.3. *Let $X_{\mathbf{a}} = \{x \in B_s : \nabla f(x) = \mathbf{a}\}$. Then $X_{\mathbf{a}}$ is convex.*

Proof. If $X_{\mathbf{a}}$ is empty or has only one element, then the statement is vacuously true. Now let $p, p' \in X_{\mathbf{a}}$. We will show that $f(X_{\mathbf{a}})$ is convex, and so by Lemma 2.24,

$$X_{\mathbf{a}} = \Pi(f(X_{\mathbf{a}}))$$

will be convex. Let $q = p + f(p)e_d$ and $q' = p' + f(p')e_d$. Let $PL_{\mathbf{a}}^q$ be the hyperplane with normal vector $(-\mathbf{a}, 1)$ and passing through q, or

$$PL^q_{\mathbf{a}} = \{y \in \mathbb{R}^d :< (y-q), (-\mathbf{a}, 1) >= 0\}.$$

Then $PL^q_{\mathbf{a}}$ is tangent to ∂P and hence all of P must lie on one side of it. Similarly, let $PL^{q'}_{\mathbf{a}}$. We claim that $PL^q_{\mathbf{a}} = PL^{q'}_{\mathbf{a}}$. Suppose not. Then, since both hyperplanes have the same normal vector, then all of P must lie between the two hyperplanes. By the previous lemma, neither plane can intersect the ball $B_{r(d)}$ because $B_{r(d)} \subset P$. Therefore, neither hyperplane may intersect the $\{x_d = 0\}$ hyperplane inside the cylinder $\mathfrak{C}_{e_d}$. We conclude that for all points inside the cylinder $\mathfrak{C}_{e_d}$ on either hyperplane, $x_d > 0$. This implies that since P lies between the two hyperplanes, that $0 \notin P$, which is a contradiction. Therefore, these hyperplanes must be equivalent, say $PL^{q'}_{\mathbf{a}} = PL^q_{\mathbf{a}} = PL_{\mathbf{a}}$, and the line between q and q' must lie on this plane. Now consider the hyperplane H in $\mathbb{R}^{d-1}$ that is perpendicular to $p - p'$. Then let $L = B_\rho(p) \cap H$. Because f is approximately differentiable at p and is also Lipschitz, this implies that f is differentiable at p and hence differentiable on L. Consider the set

$$Q = f(L).$$

Take the cone $CN = \delta_{q'} \times\!\!\!\times\, Q$ over this set with the point q' to get a surface with the same tangent plane. By convexity of the region P and the definition of f, the graph of f must lie above this cone and below the hyperplane $PL_{\mathbf{a}}$ (see Figure 3.1). Therefore, any point on the line between p and p' must have a neighborhood where the graph of the function f is between two surfaces with the same normal at the point. Therefore, f must be approximately differentiable at that point with the same derivative. $\square$

At this point, we do not know if S is relatively closed as a subset of B_s, or what the closure of S looks like. To do this, we first prove a lemma that establishes a lower

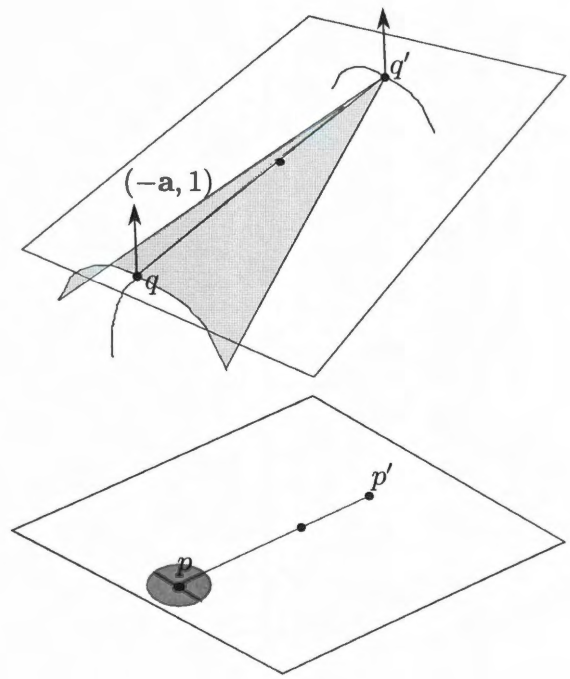

Figure 3.1: Two points on the boundary with same normal vector.

bound on the density of S at points in $\mathcal{N} = \bar{S} \cap B_s$.

Lemma 3.4. *Let $\mathcal{N} = \bar{S} \cap B_s$ and let $p \in \mathcal{N}$. Let $Q = [p - r, p + r]^{d-1}$ be the $(d-1)$-cube with sides length $2r$ around the point p when $Q \subset B_s$. Then*

$$\frac{\mathcal{H}^{n-2}(S \cap Q)}{(2r)^{d-2}} > \frac{1}{12^{d-2}(d-1)}.$$

Proof. Suppose the assumption is false. Without loss of generality, we may assume that $p = 0$. Then there exists r_0 such that $Q = [-r_0, r_0]^{d-2} \subset B_s$ and

$$\frac{\mathcal{H}^{n-2}(S \cap Q)}{(2r_0)^{d-2}} \leq \frac{1}{12^{d-2}(d-1)}.$$

Let $\pi_i : \mathbb{R}^{d-1} \to \mathbb{R}^{d-2}$ be the standard coordinate projection where

$$\pi_i(x_1, \ldots, x_{d-1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d-1})$$

for $1 \leq i \leq d-1$. Then we know that

$$\mathcal{H}^{d-2}(\pi_i(S \cap Q)) \leq \mathcal{H}^{d-2}(S \cap Q) \leq \left(\frac{r_0}{6}\right)^{d-2} \frac{1}{d-1}.$$

Let $Y \subset \mathbb{R}^{d-2}$ be such that

$$Y = \pi_1(Q) = \ldots = \pi_{d-1}(Q).$$

We then consider the decomposition of Y into thirds along each interval. We let $I_{-1} = [-r_0, \frac{-r_0}{3}]$, $I_0 = [\frac{-r_0}{3}, \frac{r_0}{3}]$, and $I_1 = [\frac{r_0}{3}, r_0]$. We then let the cubes $D_{i_1,\ldots,i_{d-2}}$ be defined as

$$D_{i_1,\ldots,i_{d-2}} = I_{i_1} \times \ldots \times I_{i_{d-2}}$$

where $i_k \in \{-1, 0, 1\}$ for $1 \leq k \leq d-2$. Note that $Y = \cup D_{i_1,\ldots,i_{d-2}}$ as i_k ranges over the possible values. Consider the "corner" cubes, those cubes $D_{i_1,\ldots,i_{d-2}}$ such that $i_k \in \{-1, 1\}$. There are 2^{d-2} such cubes. For ease of notation, we label them Y_l for $1 \leq l \leq 2^{d-2}$. We also define the appropriate translation maps $T_l : Y_l \to Y_0$ where $Y_0 = I_0 \times \ldots \times I_0$ is the middle cube. Let

$$Z = Y_0 \setminus \cup_{i,l} T_l(Y_l \cap \pi_i(S \cap Q)).$$

Since Hausdorff measure is invariant under translations, we get that

$$\begin{aligned}
\mathcal{H}^{d-2}(\cup_{i,l}T_l(Y_l \cap \pi_i(S \cap Q))) &\leq \textstyle\sum_{i,l} \mathcal{H}^{d-2}(T_l(Y_l \cap \pi_i(S \cap Q))) \\
&= \textstyle\sum_{i,l} \mathcal{H}^{d-2}(Y_l \cap \pi_i(S \cap Q) \\
&\leq \textstyle\sum_{i,l} \mathcal{H}^{d-2}(\pi_i(S \cap Q)) \\
&\leq \textstyle\sum_{i,l} \left(\frac{r_0}{6}\right)^{d-2} \frac{1}{d-1} \\
&= \left(\frac{r_0}{3}\right)^{d-2} \\
&< \left(\frac{2r_0}{3}\right)^{d-2} \\
&= \mathcal{H}^{d-2}(Y_0).
\end{aligned}$$

Therefore, $\mathcal{H}^{d-2}(Z) > 0$, so let $z = (z_1, \ldots, z_{d-2}) \in Z$, and consider the set of points:

$$\mathcal{Y} = \{(z_1 \pm \frac{r_0}{3}, z_2 \pm \frac{r_0}{3}, \ldots, z_{d-3} \pm \frac{r_0}{3}, z_{d-2} \pm \frac{r_0}{3}, z_{d-2} \pm \frac{r_0}{3})\} \subset \mathbb{R}^{d-1}$$

Where conv($\mathcal{Y}$) is a (d-1)-hypercube with side length $2r_0/3$. Let

$$L_{i,j} = \pi_i^{-1}(T_j^{-1}(z)).$$

By construction, we have $L_{i,j} \cap S = \emptyset$ and $\mathcal{Y} \subset \cup_{i,j} L_{i,j}$. We claim that $\mathcal{Y}$ is path connected in $\cup_{i,j} L_{i,j}$ and hence connected as a subset of $B_s \setminus S$.

Claim 3.5. *$\mathcal{Y}$ is path connected as a subset of $\cup_{i,j} L_{i,j}$.*

Proof. We show that we can get from any point in $\mathcal{Y}$ to the point

$$(z_1 + \frac{r_0}{3}, z_2 + \frac{r_0}{3}, \ldots, z_{d-3} + \frac{r_0}{3}, z_{d-2} + \frac{r_0}{3}, z_{d-2} + \frac{r_0}{3}).$$

We have:

$$
\begin{aligned}
& (z_1 \pm \tfrac{r_0}{3}, z_2 \pm \tfrac{r_0}{3}, \ldots, z_{d-3} \pm \tfrac{r_0}{3}, z_{d-2} \pm \tfrac{r_0}{3}, z_{d-2} \pm \tfrac{r_0}{3}) \\
\to \ & (z_1 \pm \tfrac{r_0}{3}, z_2 \pm \tfrac{r_0}{3}, \ldots, z_{d-3} \pm \tfrac{r_0}{3}, z_{d-2} \pm \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3}) \\
\to \ & (z_1 \pm \tfrac{r_0}{3}, z_2 \pm \tfrac{r_0}{3}, \ldots, z_{d-3} \pm \tfrac{r_0}{3}, z_{d-3} + \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3}) \\
\to \ & (z_1 \pm \tfrac{r_0}{3}, z_2 \pm \tfrac{r_0}{3}, \ldots, z_{d-2} + \tfrac{r_0}{3}, z_{d-3} + \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3}) \\
\to \ & \qquad\qquad\qquad \ldots \\
\to \ & (z_1 \pm \tfrac{r_0}{3}, z_1 + \tfrac{r_0}{3}, \ldots, z_{d-2} + \tfrac{r_0}{3}, z_{d-3} + \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3}) \\
\to \ & (z_1 + \tfrac{r_0}{3}, z_1 + \tfrac{r_0}{3}, \ldots, z_{d-2} + \tfrac{r_0}{3}, z_{d-3} + \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3}) \\
\to \ & (z_1 + \tfrac{r_0}{3}, z_2 + \tfrac{r_0}{3}, \ldots, z_{d-2} + \tfrac{r_0}{3}, z_{d-3} + \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3}) \\
\to \ & \qquad\qquad\qquad \ldots \\
\to \ & (z_1 + \tfrac{r_0}{3}, z_2 + \tfrac{r_0}{3}, \ldots, z_{d-3} + \tfrac{r_0}{3}, z_{d-3} + \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3}) \\
\to \ & (z_1 + \tfrac{r_0}{3}, z_2 + \tfrac{r_0}{3}, \ldots, z_{d-3} + \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3})
\end{aligned}
$$

where we travel linearly between each point along $L_{i,j}$ for some i, j. □

Thus, the set $\mathcal{Y}$ must be in the same connected component of $(B_s \backslash S)$. By Theorem 3.1, we get that

$$\nabla f(p) = \nabla f(p') = \mathbf{a}$$

for all p, p' in $\mathcal{Y}$ and some $\mathbf{a} \in \mathbb{R}^{d-1}$. Then, by the previous lemma, we know that the points in the convex hull must also be approximately differentiable and have the same gradient. Therefore,

$$
\begin{aligned}
\operatorname{conv}(\mathcal{Y}) &= [z_1 - \tfrac{r_0}{3}, z_1 + \tfrac{r_0}{3}] \times [z_2 - \tfrac{r_0}{3}, z_2 + \tfrac{r_0}{3}] \times \ldots \times \\
&\times [z_{d-3} - \tfrac{r_0}{3}, z_{d-3} + \tfrac{r_0}{3}] \times [z_{d-2} - \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3}] \\
&\times [z_{d-2} - \tfrac{r_0}{3}, z_{d-2} + \tfrac{r_0}{3}] \\
&\subset (B_s \setminus S).
\end{aligned}
$$

But 0 lies in the interior of this set and was assumed to be in the relative closure of S. This is a contradiction. □

Recall that $\Theta^k(A,x)$ is the k-dimensional density of the set $A \subset \mathbb{R}^d$ at the point x. We get the immediate corollary:

Corollary 3.6.

$$\Theta^{d-2}(S,x) \geq \frac{1}{12^{d-2}(d-1)}$$

for all $x \in \mathcal{N}$.

Theorem 3.7. $\mathcal{H}^{d-2}(\mathcal{N}) = \mathcal{H}^{d-2}(S)$.

Proof. Since S is $\mathcal{H}^{d-2}$-measurable and $\mathcal{H}^{d-2}(S) \leq (d-1)M < \infty$, we know that

$$\lim_{r\to 0} \frac{\mathcal{H}^{d-2}(B_r(x) \cap S)}{\alpha(d-2)r^{d-2}} = 0$$

For $\mathcal{H}^{d-2}$-a.e. $x \in B_s \setminus S$ [EG92]. Since this is not true for all points in $\mathcal{N}$, this implies that $\mathcal{H}^{d-2}(\mathcal{N} \setminus S) = 0$ and so $\mathcal{H}^{d-2}(\mathcal{N}) = \mathcal{H}^{d-2}(S)$. □

3.2 Partial regularity of $\sigma_{d-1}(P)$

Thus we know that $\mathcal{H}^{d-2}(\mathcal{N}) = \mathcal{H}^{d-2}(S) \leq M(d-1)$, and in particular $\mathcal{H}^{d-1}(\mathcal{N}) = 0$. Off of $\mathcal{N}$, we know that $D\frac{\partial f}{\partial x_j} = 0$ for all j and hence ∇f is piecewise constant on connected components of $B_s \setminus \mathcal{N}$. Also, those connected components must be convex by Lemma 3.3. Since f is continuous, this implies that f must be linear on $B_s \setminus \mathcal{N}$. Let $E = f(\mathcal{N})$ and let $F = f(B_s \setminus \mathcal{N})$. Recalling that this is over a particular cylinder $\mathfrak{C}_{v_n}$, when we want to make the distinction we will let $\mathcal{N}_n = \mathcal{N}$ and similarly defined for E_n and F_n. We have the following lemma:

Lemma 3.8. $F \subset \mathfrak{X}_{d-1}(P)$.

Proof. Let $q \in F$. Then there exists a point $p \in B_s \setminus \mathcal{N}$ such that $f(p) = q$. Since $B_s \setminus \mathcal{N}$ is relatively open, there exists a ball $B_\rho(p) \subset B_s \setminus \mathcal{N}$. Because this set is connected, then we know that ∇f is constant on $B_\rho(p)$ and so f is linear on $B_\rho(p)$.

Therefore, there exists a hyperplane PL such that $f(B_\rho(p)) \subset PL$. Therefore, PL is a supporting hyperplane, $q \in PL \cap \partial P$, and $\dim(PL \cap \partial P) = d-1$ by Lemma 2.44 and Corollary 2.46. Thus we get $q \in \sigma_{d-1}(P)$ and so $F \subset \mathfrak{X}_{d-1}(P)$. □

Remark 3.9. Notice that if $x \in B_s \setminus \mathcal{N}$, then $f(x)$ must lie in the relative interior of some face in $\sigma_{d-1}(P)$. Combined with Corollary 1.53, this implies that $\sigma_k(P) \cap \mathfrak{C}_{v_n} \subset f^n(\mathcal{N}_n)$ for all $k \leq d-2$, and in particular that

$$\sigma_{d-2}(P) \cap \mathfrak{C}_{v_n} \subset f^n(\mathcal{N}_n).$$

Clearly, since $\mathfrak{X}_{d-2}(P) \subset \sigma_{d-2}(P)$, we also get that

$$\mathfrak{X}_{d-2}(P) \cap \mathfrak{C}_{v_n} \subset f^n(\mathcal{N}_n).$$

Corollary 3.10 (Main Theorem 2). $\mathcal{H}^{d-2}(\partial P \setminus \mathfrak{X}_{d-1}(P)) < \infty$.

Proof. We know from the previous lemma that for our particular cylinder $\mathfrak{C}_{v_n}$, that $\mathfrak{C}_{v_n} \cap (\partial P \setminus \mathfrak{X}_{d-1}(P)) \subset E_n$. We also have that

$$\mathcal{H}^{d-2}(E_n) \leq C_1^{d-2}\mathcal{H}^{d-2}(\mathcal{N}_n) \leq C_1^{d-2}M < \infty.$$

Therefore, since our cylinders were chosen to cover ∂P and there are finitely many, we get that

$$\begin{aligned} \mathcal{H}^{d-2}(\partial P \setminus \mathfrak{X}_{d-1}(P)) &\leq \mathcal{H}^{d-2}(\cup_n \mathfrak{C}_{v_n} \cap (\partial P \setminus \mathfrak{X}_{d-1}(P))) \\ &\leq \textstyle\sum_n \mathcal{H}^{d-2}(E_n) \\ &< \infty. \end{aligned}$$

□

Because a *regular face* of dimension (d-1) must have positive (d-1)-measure, we also get the following corollary:

Corollary 3.11. *Every* regular *face of dimension (d-1) of P must be an exposed face.*

This result is actually true in general for any convex set, but it is worth noting the result specifically in this case. We also get the following corollary:

Theorem 3.12. *There are at most countably many faces in $\sigma_{d-1}(P)$.*

Proof. Let $\mathfrak{F}$ be a face in $\sigma_{d-1}(P)$. Then $\mathfrak{F}$ is convex by definition and and also $\mathrm{ri}(\mathfrak{F})$ is relatively open in ∂P. This implies that $\Pi(\mathrm{ri}(\mathfrak{F}))$ must be open in B_s and hence must contain a point with rational coordinates. Since for any two faces $\mathfrak{F}_1$ and $\mathfrak{F}_2$ we necessarily have that $\mathrm{ri}(\mathfrak{F}_1) \cap \mathrm{ri}(\mathfrak{F}_2) = \emptyset$, we know there can be at most countable many faces in $\sigma_{d-1}(P)$. ☐

3.3 Partial regularity of $\sigma_{d-2}(P)$

3.3.1 Functions of bounded hessian

We know that the sets $\mathcal{J}\frac{\partial f}{\partial x_j}$ are rectifiable, and hence there exists a vector ν_j $\mathcal{H}^{d-2}$-a.e. such that

$$D\frac{\partial f}{\partial x_j} = (\frac{\partial f}{\partial x_j}^+ - \frac{\partial f}{\partial x_j}^-)\nu_j \; \mathcal{H}^{d-2} \llcorner \mathcal{J}_{\frac{\partial f}{\partial x_j}}$$

where ν_j gives the direction of the "jump" of the function $\frac{\partial f}{\partial x_j}$. But for a point $x \in \mathcal{J}\frac{\partial f}{\partial x_j} \cap \mathcal{J}\frac{\partial f}{\partial x_k}$ we might have that $\nu_j(x) \neq \nu_k(x)$. We would like to show that this happens on a set of $\mathcal{H}^{d-2}$ measure zero.

Definition 3.13. Let $\Omega \subset \mathbb{R}^N$ be an open, connected region. Then the functions of bounded hessian is defined as

$$SBH(\Omega) := \{u \in W^{1,1}(\Omega) : \nabla u \in [SBV(\Omega)]^N\}$$

or, equivalently,

$$SBH(\Omega) := \{u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x^j} \in SBV(\Omega) \text{ for all } 1 \leq j \leq N\}.$$

From this definition, we arrive at the following conclusion:

Lemma 3.14. $f \in SBH(B_s)$.

Proof. We know that f is Lipschitz and hence $f \in W^{1,1}(B_s)$. Also, by Theorem 2.41, we know that $\frac{\partial f}{\partial x_j} \in SBV(B_s)$ for $1 \leq j \leq (d-1)$. □

For a function $u \in SBH(\Omega)$, we associate to it the vector-valued measure D^2u such that

$$(D^2u)_j = D\frac{\partial u}{\partial x_j}$$

or

$$D^2u = (D\frac{\partial u}{\partial x_1}, D\frac{\partial u}{\partial x_2}, \ldots, D\frac{\partial u}{\partial x_N}).$$

Thus we may represent D^2u in matrix form as

$$D^2u = \nabla\nabla u\mathcal{L}^N + [\nabla f^+ - \nabla f^-](\nu_{\nabla u})\mathcal{H}^{N-1} \llcorner \mathcal{J}u$$

where

$$(\nabla\nabla u)_{ij} = (\nabla(\nabla f)_i)_j$$

$$[\nabla f^+ - \nabla f^-]_{ij} = \delta_{ij}((\nabla f)_j^+ - (\nabla f)_j^-)$$

$$(\nu_{\nabla u})_{ij} = (\nu_i)_j$$

and

$$\mathcal{J}u = \cup_{i=1}^N \mathcal{J}((\nabla u)_i).$$

D^2u is a symmetric hessian measure and we have the following theorem:

Theorem 3.15 ([AFP00] §3.9,[Alb93]). *If $u \in [BV(\Omega)]^m$ and $Du = g|Du|$ then g has rank one for $|D^j u| + |D^c u|$-a.e. point of Ω.*

We get the following corollary:

Corollary 3.16. *There exists a set $\mathcal{G} \subset \mathcal{N}$ such that $\mathcal{H}^{d-2}(\mathcal{N} \setminus \mathcal{G}) = 0$ and $\nu_j(x) = \pm\nu_k(x)$ for all $1 \leq j, k \leq (d-1)$ and all $x \in \mathcal{G}$.*

Proof. Since $|\nabla f| \in [SBV(\Omega)]^{d-1}$, we know that $|D^c \nabla f| = 0$ and so we let $\mathcal{G} \subset S$ be the set where $\nu_{\nabla f}$ has rank one. From Theorem 3.15 we get that $\mathcal{H}^{d-2}(S \setminus \mathcal{G}) = 0$ which implies $\mathcal{H}^{d-2}(\mathcal{N} \setminus \mathcal{G}) = 0$. □

Now we would like to show that $\mathcal{H}^{d-2}$ almost all the points in $\partial P \setminus \sigma_{d-1}(P)$ are elements of $\mathfrak{X}_{d-2}(P)$. We define the function $\eta_\rho : \partial P \to \mathbb{N}$ by

$$\eta_\rho(q) = \#\{\text{faces } H \in \sigma_{d-1}(P) : H \cap B_\rho(q) \neq \emptyset\}.$$

Notice that η_ρ is monotonic in ρ and thus we define $\eta : \partial P \to \mathbb{N}$ as

$$\eta(q) = \lim_{\rho \to 0} \eta_\rho(q).$$

Claim 3.17. $q \in f(\mathcal{G}) \Rightarrow \eta(q) = 2$.

Proof. Assume $\eta(q) \neq 2$. We know that $\mathcal{H}^{d-1}(\partial P \setminus \sigma_{d-1}(P)) = 0$. Therefore, $\eta_\rho(q) \geq 1$ because $B_\rho(q) \cap \partial P$ necessarily has positive $\mathcal{H}^{d-1}$ measure.

Case 1: Suppose that $\eta(q) = 1$. Then we may find a ball $B_{\rho_0}(q)$ such that $\mathcal{H}^{d-1}$-a.e. point in B_{ρ_0} is in some face H_0. Since this face is necessarily convex, this implies that $q \in \mathrm{ri}(H_0)$ and thus $q \notin f(\mathcal{G})$.

Case 2: Suppose that $\eta(q) \geq 3$. Thus, for every ball $B_\rho(q)$ there must be at least three faces H_1, H_2, and H_3. By convexity of P and previous arguments, no two faces may have the same normal vector. Therefore, for any pair of faces, we may find a sets A_1 and A_2 of positive measure in $\Pi(B_\rho(q))$ where $\frac{\partial f}{\partial x_j}|_{A_1} \neq \frac{\partial f}{\partial x_j}|_{A_2}$.

Case (i): There exist three faces H_1, H_2, H_3 such that $H_i \cap B_\rho(x) \neq \emptyset$ for all $\rho > 0$. Let $\mathbf{a}_i$ be the normal vector to the supporting plane for the face H_i. Then there exists k, l such that $(\mathbf{a}_1)_k \neq (\mathbf{a}_2)_k$ and $(\mathbf{a}_2)_l \neq (\mathbf{a}_3)_l$. We may consider the projections down $\Pi(B_\rho(q) \cap H_i)$. Each of these regions are convex (and hence connected), and

$$\frac{\partial f}{\partial x_k}|_{\Pi(B_\rho(q)\cap H_1)} \neq \frac{\partial f}{\partial x_k}|_{\Pi(B_\rho(q)\cap H_2)}$$

and similarly

$$\frac{\partial f}{\partial x_l}|_{\Pi(B_\rho(q)\cap H_2)} \neq \frac{\partial f}{\partial x_l}|_{\Pi(B_\rho(q)\cap H_3)}.$$

Assume $\Pi(q) \in \mathcal{J}\nabla f$, (if not, then $\Pi(q) \notin \mathcal{G}$ and we are done). Then there exists $(d-2)$-hyperplanes PL_k and PL_l that separate $B_\rho \cap \Pi(H_1)$ from $B_\rho \cap \Pi(H_2)$ and $B_\rho \cap \Pi(H_2)$ from $B_\rho \cap \Pi(H_3)$, respectively. Since the $(d-2)$-hyperplanes are separated by sets of positive measure, we know that for whatever hyperplanes we pick, $PL_k \neq PL_l$. Since the hyperplanes with normals defined by $\nu_k(\Pi(q))$ and $\nu_l(\Pi(q))$ satisfy this, then we conclude that $\nu_k(\Pi(q)) \neq \nu_l(\Pi(q))$ and hence $\Pi(q) \notin \mathcal{G}$.

Case (ii). Now suppose that there do not exist such planes that occurred in *Case (i)*. Since for every ρ we may find 3 different supporting hyperplanes, we conclude that there exists a sequence of faces $\{H_i\}_{i=1}^{\infty}$ such that $B_{\rho_i}(q) \cap H_i \neq \emptyset$ and $\rho_i \to 0$, with supporting hyperplanes and associated normal vectors $\mathbf{a}_i$ such that $\mathbf{a}_i \neq \mathbf{a}_j$ if $i \neq j$. We thus conclude there exists a subsequence (relabeled) and an index k_0 such that $(\mathbf{a}_i)_{k_0} \neq (\mathbf{a}_j)_{k_0}$ if $i \neq j$. Thus, for every ρ_i, there exist convex, connected sets of positive measure $A_i \subset B_{\rho_i}$ such that $\frac{\partial f}{\partial x_{k_0}}|_{A_i} \neq \frac{\partial f}{\partial x_{k_0}}|_{A_j}$. Therefore, $\Pi(q) \notin \mathcal{J}\nabla f$ and hence $q \notin f(\mathcal{G})$. $\square$

Theorem 3.18. $f(\mathcal{G}) \subset \mathfrak{X}_{d-2}(P)$.

Proof. Let $x \in \mathcal{G}$. Since $\mathcal{J}\nabla f$ is rectifiable at x and has an approximate tangent plane $\nu_{\nabla f}^{\perp}$, we know by the preceding claim that there exists a neighborhood of x, $B_\rho(x) \subset B_s$ such that $B_\rho(x)$ only contains the images of two faces, say H_1 and H_2. Consider the regions $D_i = \Pi(H_i) \cap B_\rho(x)$ for $i = 1, 2$. We know these regions are convex, and $S \cap B_\rho(x) \subset (D_1 \cap D_2)$, and therefore is a subset of a convex set. Since S must necessarily separate regions D_1 and D_2, this implies that $S \cap B_\rho(x) = \nu_{\nabla f}^{\perp} \cap B_\rho(x)$. Therefore, $S \cap B_\rho(x) \subset \mathcal{G}$ and $\dim(\mathcal{G} \cap B_\rho(x)) = d-2$. By Lemma 2.25 and Corollary 2.27 we get that $x \in \mathfrak{X}_{d-2}(P)$. $\square$

Corollary 3.19 (Main Theorem 3). $\mathcal{H}^{d-2}(\partial P \setminus (\mathfrak{X}_{d-1}(P) \cup \mathfrak{X}_{d-2}(P))) = 0$.

Proof. By the previous theorem, for each cylinder $\mathfrak{C}_{v_n}$ we have that

$$(\partial P \cap \mathfrak{C}_{v_n}) \setminus (\mathfrak{X}_{d-1}(P) \cup \mathfrak{X}_{d-2}(P)) \subset (\mathcal{N}_n \setminus \mathcal{G}_n).$$

Thus,

$$\begin{aligned}
\mathcal{H}^{d-2}(\partial P \setminus (\mathfrak{X}_{d-1}(P) \cup \mathfrak{X}_{d-2}(P))) &= \mathcal{H}^{d-2}(\cup_n(\partial P \cap \mathfrak{C}_{v_n}) \setminus (\mathfrak{X}_{d-1}(P) \cup \mathfrak{X}_{d-2}(P)) \\
&\leq \textstyle\sum_n \mathcal{H}^{d-2}((\partial P \cap \mathfrak{C}_{v_n}) \setminus (\mathfrak{X}_{d-1}(P) \cup \mathfrak{X}_{d-2}(P))) \\
&\leq \textstyle\sum_n \mathcal{H}^{d-2}(f(\mathcal{N}_n \setminus \mathcal{G}_n)) \\
&= 0.
\end{aligned}$$

$\square$

Again, since any *regular* face of dimension (d-2) must have positive measure, we obtain:

Corollary 3.20. *Every* regular *face of dimension (d-2) of P must be an exposed face.*

Chapter 4

Lower semicontinuity

Finally, we would like to show the $\mathcal{H}^{d-2}$ measure of the edge set for the limiting convex set is less than or equal to the $\mathcal{H}^{d-2}$ measure of the edge set for any polytope with volume one. This result is significant because from Chapter 3 we know that $\mathcal{H}^{d-2}$ almost all the points in the boundary of the limiting set are either face points or edge points. Otherwise, our limiting object could have a "curved" boundary like a closed ball and $\zeta_{d-2}(\mathbb{B}^d) = 0$. This is our Main Theorem 4 and is found in §4.4

4.1 Spherical representations

We now wish to consider the boundaries of the polyhedra as graphs of functions over the sphere. We note again that the isoperimetric ratios are invariant under scaling. Hence, in this section we will be implicitly scaling our space so that we may assume that $r(d) = 1$, and so we will let $R_2(d) = \frac{R(d)}{r(d)}$. This will make notation simpler. Our goal is to show the lower semicontinuity of the total jump sets and hence for the generalized edge sets of our limiting polyhedron. Let P be a closed convex set that contains S_1. Then P is in particular star convex around the origin. Let $x \in S_1$. Then there exists a maximum number $u(x) \geq 1$ such that

$$u(x)x \in P.$$

Thus, we get that

$$P = \{tu(x)x : t \in [0,1], x \in S_1\}$$

and

$$\partial P = \{u(x)x : x \in S_1\}.$$

4.1.1 Lipschitz estimates and convergence

We let $u : S_1 \to \mathbb{R}$ be the spherical function for P and $u_i : S_1 \to \mathbb{R}$ be the spherical function for P_i. Note that since $P, P_i \subset \bar{B}_{R_2(d)}$, then we get that $1 \leq u_i(x), u(x) \leq R_2(d)$. Again, we would like to show that u, u_i are Lipschitz. First, we prove a useful lemma:

Lemma 4.1. $(1- < x, y >) = \frac{1}{2}||x-y||^2$ *for* $x, y \in S_1$.

Proof. Well,

$$\begin{aligned} ||x-y||^2 &= < x-y, x-y > \\ &= < x,x > + < y,y > -2 < x,y > \\ &= 2(1- < x,y >). \end{aligned}$$

□

Theorem 4.2. *There exists constant* C_4 *such that*

$$|u(x) - u(y)| \leq C_4||x-y||$$

and

$$|u_i(x) - u_i(y)| \leq C_4||x-y||$$

for all $x, y \in S_1$ *and all* $i > 0$.

Proof. We prove for u. Let $x, y \in S_1$. Let PL be the 2-plane defined by x, y and 0 and consider the cone

$$CN = \delta_{u(x)x} \times\!\!\times B_{\frac{1}{2}}.$$

Then we know by convexity that $PL \cap CN \subset P$. Assume without loss of generality that $u(x) \geq u(y)$.

Case 1: Assume $\frac{1}{2} \leq < x, y >$. Let α denote the angle between x and y and let $\cos(\beta) = \frac{1}{2u(x)}$. Note that since $1 \leq u(x) \leq R_2(d)$, we get that

$$\frac{1}{2R_2(d)} \leq \cos(\beta) \leq \frac{1}{2}.$$

Therefore, we get a lower bound on β, and

$$\alpha \leq \frac{\pi}{3} \leq \beta.$$

By convexity (see Figure 4.1), we know that $py \in P$, and so $u(y) \geq p$ by definition and $\cos(\beta - \alpha) = \frac{1}{2p}$, or

$$p = \frac{1}{2\cos(\beta - \alpha)} = \frac{1}{2(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta))}.$$

Then we find

$$
\begin{aligned}
\frac{(u(x)-u(y))^2}{||x-y||^2} &\leq \frac{(u(x)-p)^2}{||x-y||^2} \\
&= \frac{(2u(x)\cos(\alpha)\cos(\beta)+2u(x)\sin(\alpha)\sin(\beta)-1)^2}{8(\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta))^2(1-\cos(\alpha))} \\
&= \frac{(\cos(\alpha)-1+2u(x)\sin(\alpha)\sin(\beta))^2}{8(\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta))^2(1-\cos(\alpha))} \\
&\leq \frac{(u^2(x)-\frac{1}{4})\sin^2(\alpha)}{2(\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta))^2(1-\cos(\alpha))} \\
&\leq \frac{R_2^2(d)(1-\cos^2(\alpha))}{2(\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta))^2(1-\cos(\alpha))} \\
&\leq \frac{R_2^2(d)(1+\cos(\alpha))}{2(\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta))} \\
&\leq \frac{R_2^2(d)}{\cos(\alpha)\cos(\beta)} \\
&\leq 4R_2^3(d).
\end{aligned}
$$

Therefore, if $\frac{1}{2} \leq < x, y >$, then

$$|u(x) - u(y)| \leq 2R_2^{\frac{3}{2}}(d)||x - y||.$$

Case 2. Now suppose that $< x, y > \leq \frac{1}{2}$. Then

$$||x - y||^2 = 2(1 - < x, y >) \geq 1$$

and so $||x - y|| \geq 1$. Thus, when $\frac{1}{2} \leq < x, y >$, we get

$$|u(x) - u(y)| \leq 2R_2(d) \leq 2R_2(d)||x - y||.$$

Since we know $R_2(d) \geq 1$, we know that $2R_2(d) \leq 2R_2(d)^{\frac{3}{2}}$. We may therefore set

$$C_4 := 2R_2(d)^{\frac{3}{2}}$$

and we get our desired conclusion for all $x, y \in S_1$. □

Also, since $\partial P_i \to \partial P$, we again get the uniform convergence $u_i \to u$. To do this,

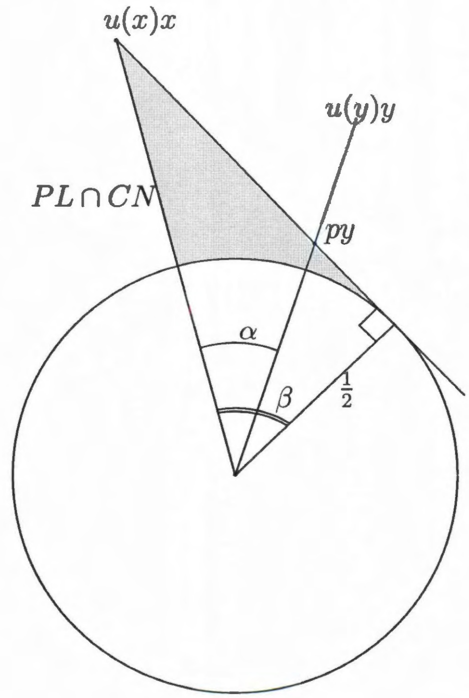

Figure 4.1: 2-plane PL intersected with the cone from $u(x)x$ with $B_{\frac{1}{2}}$.

we first prove a couple of preliminary lemmas:

Lemma 4.3. *Let* $1 \leq a, b$ *and* $x, y \in S_1$. *Then*

$$||ax - by|| \geq \frac{1}{\sqrt{2}}||x - y||.$$

Proof. Case 1. Suppose $< x, y > \leq 0$. Then

$$\begin{aligned} ||ax - by||^2 &= < ax - by, ax - by > \\ &= a^2 < x, x > + b^2 < y, y > -2ab < x, y > \\ &\geq < x, x > + < y, y > -2 < x, y > \\ &= ||x - y||^2. \end{aligned}$$

Case 2. Suppose $< x, y > \geq 0$. Then

$$\begin{aligned} ||ax - by||^2 &\geq ||ax - a < x, y > y||^2 \\ &= a^2(< x, x > + < x, y >^2 < y, y > -2 < x, y >^2) \\ &= a^2(1- < x, y >^2) \\ &= a^2(1- < x, y >)(1+ < x, y >) \\ &\geq \tfrac{1}{2}||x - y||^2. \end{aligned}$$

□

Lemma 4.4. *Let $x, y \in S_1$. Then*

$$||ax - by|| \geq |a - b|.$$

Proof. We have

$$\begin{aligned} ||ax - by||^2 &= < ax - by, ax - by > \\ &= a^2 < x, x > +b^2 < y, y > -2ab < x, y > \\ &= a^2 + b^2 - 2ab < x, y > \\ &\geq a^2 + b^2 - 2ab||x|| \, ||y|| \\ &= (a - b)^2. \end{aligned}$$

□

Theorem 4.5. *$u_i \to u$ in $L^\infty(S_1)$.*

Proof. Let $x \in S_{r(d)}$. Let $\epsilon > 0$ and let $\delta = \epsilon(1 + \sqrt{2}R_2(d) + \sqrt{2}C_4)^{-1}$. Then there exists N such that $i > N$ implies $d_H(\partial P_i, \partial P) \leq \delta$. Thus, there exists $y \in S_1$ such that $d(u_i(y)y, u(x)x) \leq \delta$. By the previous lemmas, we know that

$$||u(x)x - u_i(y)y|| \geq \frac{1}{\sqrt{2}}||x - y||$$

and

$$||u(x)x - u_i(y)y|| \geq |u(x) - u_i(y)|$$

and so we get that $||x - y|| \leq \sqrt{2}\delta$ and $|u(x) - u_i(y)| \leq \delta$. Therefore, if $i > N$,

$$\begin{aligned}
|u(x) - u_i(x)| &= ||u(x)x - u_i(x)x|| \\
&\leq ||u(x)x - u_i(y)y|| + ||u_i(x)x - u_i(y)y|| \\
&\leq ||u(x)x - u_i(y)y|| + ||u_i(x)x - u_i(x)y|| + ||u_i(x)y - u_i(y)y|| \\
&= ||u(x)x - u_i(y)y|| + u_i(x)||x - y|| + |u_i(x) - u_i(y)|||y|| \\
&\leq \delta + R_2(d)\sqrt{2}\delta + C_4||x - y|| \\
&\leq \delta(1 + \sqrt{2}R_2(d) + \sqrt{2}C_4) \\
&\leq \epsilon.
\end{aligned}$$

□

4.2 Face convergence

We now want to view the faces of the polytopes as currents and show that they converge in the flat metric. To do this, we will need some preliminary definitions.

4.2.1 BV functions on a manifold

BV functions can be equivalently defined on a manifold using tangent vector fields.

Definition 4.6. Let M be a manifold and $u \in L^1(M)$. We say that u is a *function of bounded variation on* M ($u \in BV(M)$) if

$$|Du|(M) := \sup\{\int_M u \operatorname{div}\omega \, d\mu : \omega \in \Gamma_c(T^*M), |\omega| \leq 1\} < \infty$$

and note that if M is compact, then $\Gamma_c(T^*M) = \Gamma_0(T^*M) = \Gamma(T^*M)$. One obtains the usual results and theorems for $BV(M)$ as we have for $BV(\Omega)$.

4.2.2 Face partitions and currents

Now let $\Pi_r : \mathbb{R}^d \to S_1$ be defined as

$$\Pi_r(x) = \frac{x}{||x||}$$

and consider the sets

$$D_i := \Pi_r(\sigma_{d-2}(P_i)).$$

Also, let F_i^j be the finite set of faces of $\sigma_{d-1}(P_i) = \cup_j F_i^j$, and define

$$K_i^j := \Pi_r(F_i^j).$$

Then we know that by the homeomorphism between P_i and S^1 that $\cup_j \mathrm{rb}(F_i^j) = D_i$ and, therefore,

$$\mathcal{H}^{d-2}(\cup_j \mathrm{rb}(F_i^j)) = \mathcal{H}^{d-2}(D_i) \leq \mathcal{H}^{d-2}(\sigma_{d-2}(P_i)) \leq M_2$$

where the $\mathrm{rb}(F_i^j)$ is obtained by viewing $F_i^j \subset S^1$. Since F_i^j is convex for each j, it must therefore be path-connected and hence connected. Since we have a homeomorphism between F_i^j and K_i^j for each j, we know that K_i^j must also be connected for all j. Now, let

$$A_i^1 := K_i^1$$

and

$$A_i^{j+1} = K_i^{j+1} \setminus (\cup_{k=1}^j K_i^k)$$

in the standard way to get $\{A_i^j\}$ to be pairwise disjoint. Also, because $\Pi_r(\partial P_i) = S_1$, we get that $\cup_j A_i^j = S_1$ and so $\{A_i^j\}$ is a partition of S_1. Now, for each i, relabel A_i^j

(and correspondingly F_i^j and K_i^j) so that

$$\mathcal{H}^{d-1}(A_i^j) \leq \mathcal{H}^{d-1}(A_i^{k+1}).$$

This is possible because, for each i, there are finitely many A_i^j and $\mathcal{H}^{d-1}(S^1) < \infty$. Also note that $\mathrm{rb}(A_i^j) = \mathrm{rb}(K_i^j)$ as subsets of S_1. By Theorem 1.51, we know that for $\mathcal{H}^{d-2}$ almost all points $x \in \sigma_{d-2}(P_i)$, there are exactly two faces $F_i^{j_1}$ and $F_i^{j_2}$ such that $x \in \mathrm{rb}(F_i^{j_1}) \cap \mathrm{rb}(F_i^{j_2})$. Therefore, we get

$$\sum_j P(A_i^j, S_1) = \sum_j \mathcal{H}^{d-2}(\mathrm{rb}(A_i^j)) = 2\mathcal{H}^{d-2}(D_i) \leq 2M_2.$$

Thus, we have the following result:

Lemma 4.7. *For every i, the set $\{A_i^j\}$ is a finite, ordered Caccioppoli partition of S^1, and*

$$\sum_j P(A_i^j, S^1) \leq 2M_2.$$

By Theorem 2.44, we get arrive at the following conclusion:

Corollary 4.8. *There exists partition $\{A^j\}$ of S_1 and a subsequence $i(k)$ such that $A_{i(k)}^j \to_{k\to\infty} A^j$ in measure.*

Again, relabel indices for the subsequence. Then the partitions have the following lower-semicontinuity property:

Lemma 4.9. $P(A^j, S_1) \leq \liminf_{i\to\infty} P(A_i^j, S_1)$.

Proof. We know that $A_i^j \to A^j$ in measure if and only if $\chi_{A_i^j} \to \chi_{A^j}$ in $L^1(S_1)$. Because the total variation is lower semicontinuous with respect the the L^1 norm, we have

$$P(A^j, S_1) = |D\chi_{A^j}|(S_1) \leq \liminf |D\chi_{A_i^j}|(S_1) = \liminf_{i\to\infty} P(A_i^j, S_1).$$

$\square$

4.3 Convergence of face currents

In this section, we will show the faces of each of the polytopes, when viewed as currents, converge in the flat metric.

Let ξ be a unit orientation for the sphere and consider the $(d-1)$-currents defined by

$$\mathbf{A}^j(\phi) := \int_{S_1} <\phi, \xi> d\mathcal{H}^{d-1} \llcorner A^j$$

and

$$\mathbf{A}_i^j(\phi) := \int_{S_1} <\phi, \xi> d\mathcal{H}^{d-1} \llcorner A_i^j$$

for all $\phi \in \Omega^{d-1}(S_1)$ (the set of all $(d-1)$-forms on S_1). Now consider the sets $\{T^j\}$ and $\{T_i^j\}$ such that

$$T^j := u(A^j)$$

and

$$T_i^j := u_i(A_i^j)$$

and, remembering that $u, u_i : S^1 \to \mathbb{R}$ are Lipschitz, their corresponding (d-1)-currents:

$$\mathbf{T}^j := u^\sharp \mathbf{A}^j$$

and

$$\mathbf{T}_i^j := u_i^\sharp \mathbf{A}_i^j.$$

Let $\mathcal{F}$ be the usual flat norm on the space of currents. We wish to prove the following lemma:

Lemma 4.10. $\mathcal{F}(\mathbf{T}_i^j - \mathbf{T}^j) \to_{i\to\infty} 0$.

Proof. Let $\epsilon > 0$. Let $\delta = \epsilon(\mathcal{H}^{d-1}(S_1) + M_2 + C_4)^{-1}$. Since $u_i \to u$ in $L^\infty(S_1)$, there exists N_1 such that $i \geq N_1$ implies $||u - u_i||_\infty \leq \delta$. Also, there exists N_2 such that

$i \geq N_2$ implies $\mathcal{H}^{d-1}(A_i^j \triangle A^j) \leq \delta$. Let $N = \max\{N_1, N_2\}$ and assume $i \geq N$. Let $\mathbf{R} = u^\sharp(\mathbf{A}_i^j)$. Then

$$\mathcal{F}(\mathbf{R} - \mathbf{T_j}) \leq \mathbb{M}(\mathbf{R} - \mathbf{T_j}) \leq C_4 \mathcal{H}^{d-2}(A_i^j \triangle A^j) \leq C_4 \delta$$

Now define the current $\mathbf{Q}$ by defining the set

$$Q = \{tx \in \mathbb{R}^d : x \in A_i^j, \min\{u(x), u_i(x)\} \leq t \leq \max\{u(x), u_i(x)\}\}$$

Q is clearly rectifiable, and hence we get the d-current defined by

$$\mathbf{Q} = \int_{S_1} < \phi, \zeta > \mathcal{H}^d \llcorner Q.$$

and, by the coarea formula,

$$\mathbb{M}(\mathbf{Q}) = \mathcal{H}^d(Q) \leq \int_Q [J\Pi_r]\mathcal{H}^d = \int_{S_1} |u(x) - u_i(x)| d\mathcal{H}^{d-1} \llcorner A_i^j \leq \mathcal{H}^{d-1}(S_1)\delta.$$

Finally, let X be the set defined by

$$X := \{tx : x \in \partial^* A_i^j = \mathrm{rb}(A_i^j) \subset S_1, \min\{u(x), u_i(x)\} \leq t \leq \max\{u(x), u_i(x)\}\}$$

and, again, let $\mathbf{X}$ be its associated (d-1)-current with appropriate orientation. Then, similarly,

$$\mathbb{M}(\mathbf{X}) \leq \int_{S_1} |u(x) - u_i(x)| \mathcal{H}^{d-2} \llcorner \partial^* A_i^j \leq \delta P(A_i^j, S_1) \leq \delta M_2.$$

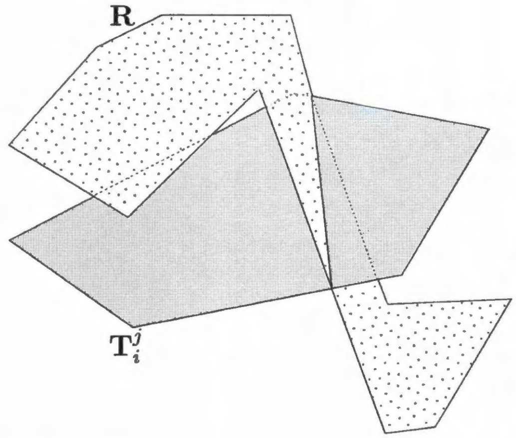

Figure 4.2: Possible configuration of the currents $\mathbf{T}_i^j$ and $\mathbf{R}$.

Therefore, since $\mathbf{T}_i^j - \mathbf{R} = \partial\mathbf{Q} + \mathbf{X}$, we get that

$$\begin{aligned}\mathcal{F}(\mathbf{T}_i^j - \mathbf{T}^j) &\leq \mathcal{F}(\mathbf{T}_i^j - R) + \mathcal{F}(\mathbf{R} - \mathbf{T}^j) \\ &\leq \mathbb{M}(\mathbf{Q}) + \mathbb{M}(\mathbf{X}) + \mathbb{M}(\mathbf{R} - \mathbf{T}_i^j) \\ &\leq \mathcal{H}^{d-1}(S_1)\delta + M_2\delta + C_4\delta \\ &\leq \epsilon.\end{aligned}$$

□

Hence we obtain the lower semicontinuity on the masses of the boundaries:

Corollary 4.11. $\mathbb{M}(\partial\mathbf{T}^j) \leq \liminf_{i\to\infty} \mathbb{M}(\partial\mathbf{T}_i^j)$.

Proof. Since $\mathbf{T}_i^j \to \mathbf{T}^j$ in the flat norm, we also get that $\mathbf{T}_i^j \rightharpoonup \mathbf{T}^j$ weakly* (or weakly), which implies $\partial\mathbf{T}_i^j \rightharpoonup \partial\mathbf{T}^j$ weakly*. Thus, lower semicontinuity of masses follows. □

But we also know that

$$\mathbb{M}(\partial\mathbf{T}_i^j) = \mathcal{H}^{d-2}(u_i(\partial^* A_i^j)) = \mathcal{H}^{d-2}(\mathrm{rb}(F_i^j))$$

and

$$\sum_j \mathcal{H}^{d-2}(\text{rb}F_i^j) = 2\zeta_{d-2}(P_i).$$

Thus we conclude with the following lemma:

Lemma 4.12. $\mathcal{H}^{d-2}(u(\cup_j \partial^* A^j)) \leq \liminf_{i\to\infty} \zeta_{d-2}(P_i)$.

Proof. By the properties of Caccioppoli partitions and $\text{supp}(\partial \mathbf{T}_i^j) = \text{rb}(F_i^j)$, we find:

$$\begin{aligned}
\mathcal{H}^{d-2}(u(\cup_j \partial^* A^j) &= \tfrac{1}{2}\textstyle\sum_j \mathcal{H}^{d-2}(u(\partial^* A^j)) \\
&= \tfrac{1}{2}\textstyle\sum_j \mathbb{M}(\partial \mathbf{T}^j) \\
&\leq \tfrac{1}{2}\textstyle\sum_j \liminf \mathbb{M}(\partial \mathbf{T}_i^j) \\
&\leq \tfrac{1}{2}\liminf \textstyle\sum_j \mathbb{M}(\partial \mathbf{T}_i^j) \\
&= \tfrac{1}{2}\liminf \textstyle\sum_j \mathcal{H}^{d-2}(\text{rb}(F_i^j)) \\
&= \liminf \zeta_{d-2}(P_i).
\end{aligned}$$

□

4.4 Lower semicontinuity

Now we want to show that ζ_{d-2} to be lower-semicontinuous so that the $\mathcal{H}^{d-2}$ measure of the "edge set" for the limiting convex set will be less than or equal to the $\mathcal{H}^{d-2}$ measure of the edge set of any polytope with unit volume. We first prove the following lemma:

Lemma 4.13. *$f^n(\mathcal{N}_n) \subset u(\cup_j \partial^* A^j)$ for all n.*

Proof. Let $p \in \partial P$ such that $p \notin u(\cup_j \partial^* A^j)$. Since A^j was a partition of the sphere, there exists j_0 such that $p \in u(A^{j_0})$. Since $A_i^{j_0} \to A^{j_0}$ in measure and A_i^j were convex, we get that A^{j_0} must be convex, and because $\Pi_r(p) \in A^{j_0} \setminus \partial^* A^{j_0}$, we necessarily know that $\Pi_r(p) \in \text{ri}(A^{j_0})$. Then there exists a ball $B_\rho(\Pi_r(p))$ such that $B_\rho(\Pi_r(p)) \subset\subset$

$\mathrm{ri}(A^{jo})$. Since the A_i^{jo} are, again, convex, we know that for large enough i,

$$\Pi_r(\sigma_{d-2}(P_i) \cap B_\rho(\Pi_r(p)) = \emptyset.$$

Therefore, since projection outside the ball is a contraction, we know that

$$B_\rho(p) \cap \sigma_{d-2}(P_i) = \emptyset$$

for large enough i. Pick a cylinder $\mathfrak{C}_{v_n}$ such that $p \in \mathfrak{C}_{v_n}$ and, if necessary, make ρ smaller so that $B_\rho(p) \subset \mathfrak{C}_{v_n}$. Then for large enough i,

$$\Pi_{v_n}(B_\rho(p) \cap \sigma_{d-2}(P_i)) = \emptyset.$$

Therefore, for large enough i, we get that $D\nabla f_i^n = 0$ on $\Pi_{v_n}(B_\rho(p))$ and thus, by the weak convergence of measures, $D\nabla f^n = 0$ on $\Pi_{v_n}(B_\rho(p))$. Therefore, $\Pi_{v_n}(p) \notin \mathcal{N}$ and so $p \notin f^n(\mathcal{N}_n)$. $\square$

This leads to our final result:

Theorem 4.14 (Main Theorem 4). $\zeta_{d-2}(P) \leq \liminf \zeta_{d-2}(P_i)$.

Proof. Combining Remark 3.9 and the previous theorems, we conclude:

$$\begin{aligned}
\zeta_{d-2}(P) &= \mathcal{H}^{d-2}(\sigma_{d-2}(P)) \\
&\leq \mathcal{H}^{d-2}(\cup_n f^n(\mathcal{N}_n)) \text{ (by Remark 3.9)} \\
&\leq \mathcal{H}^{d-2}(u(\cup_j \partial^* A^j)) \text{ (by Lemma 4.13)} \\
&\leq \liminf \zeta_{d-2}(P_i) \text{ (by Lemma 4.12)}
\end{aligned}$$

$\square$

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